Chapter 1

Angles and Triangles

1.1 Angles and Angle Measure

Angles and Rotation An **angle** is the geometric configuration formed by two rays sharing a common endpoint. This endpoint is called the **vertex** of the angle.

The measure of an angle, commonly denoted θ, is defined by designating one ray as the **initial ray** and the other as the **terminal ray**. The angle measure θ is the amount of rotation of the terminal ray about the vertex starting from the initial ray. By convention, counterclockwise rotation is positive and clockwise rotation is negative.

Degrees and Radians Rotation is measured in units of **degrees** (°) or **radians** (rad).

Degrees are defined by assigning a complete rotation a measure of 360°.

Radians are defined as the ratio of the length s of the circular arc traced by a point on the terminal ray at a fixed distance r from the vertex, as the terminal ray rotates through the angle θ, to the distance r:

$$\theta \,\mathrm{rad} = \frac{s}{r}$$

For a complete rotation, the traced path is a circle of radius r centered at the vertex. Since the circumference of a circle is $2\pi r$, the radian measure of a complete rotation is 2π.

This establishes the relationship between degrees and radians:

$$360^\circ = 2\pi$$

$$180^\circ = \pi$$

To convert from radians to degrees, multiply by $\frac{180}{\pi}$. To convert from degrees to radians, multiply by $\frac{\pi}{180}$.

Standard Position and Coterminal Angles An angle in the coordinate plane is said to be in **standard position** if its vertex is at the origin and its initial ray lies along the positive x-axis.

Two angles are **coterminal** if their terminal rays coincide when both angles are placed in standard position. Coterminal angles differ by an integer multiple of a complete rotation.

For example, 90° and 810° are coterminal since they differ by 720°, which is an integer multiple of 360°. Similarly, $\frac{\pi}{2}$ rad and $\frac{9\pi}{2}$ rad are coterminal since they differ by 4π, which is an integer multiple of 2π.

For a set of coterminal angles, the **principal angle** is defined as the unique angle in the interval $[0^\circ, 360^\circ)$.

Classification of Angles Angles are classified by their measures as follows:

Name	**Measure**
Acute	$0^\circ < \theta < 90^\circ$
Right	$\theta = 90^\circ$
Obtuse	$90^\circ < \theta < 180^\circ$
Straight	$\theta = 180^\circ$
Reflex	$180^\circ < \theta < 360^\circ$

1.2 Triangles

Definition A **triangle** is a polygon with three sides and three vertices. The measures of the interior angles of a triangle always sum to 180°.

Triangles are classified by their angle measures and their side lengths.

Classification by Angles Triangles are classified by angles as:

Acute triangle	All interior angles are acute
Right triangle	One interior angle is right and the other two are acute
Obtuse triangle	One interior angle is obtuse and the other two are acute

Right Triangle Terminology In a right triangle, the side opposite the right angle is the longest side and is called the **hypotenuse**. The two sides are called the **legs**. With respect to a given acute angle θ in a right triangle, the two legs are referred to as the **adjacent** side and the **opposite** side.

Classification by Side Lengths Triangles are classified by side lengths as:

Equilateral triangle	All three sides are equal
Isosceles triangle	Two sides are equal
Scalene triangle	All three sides are different

Equal sides of a triangle are always opposite equal angles. In an equilateral triangle, all interior angles measure 60°. In an isosceles triangle, the two angles opposite the equal sides are equal. In a scalene triangle, all interior angles are different.

Chapter 2

Right Triangle Method

2.1 Definition of the Trigonometric Functions

The Trigonometric Functions The measure of an angle gives the amount of rotation from the initial ray to the terminal ray. The **trigonometric functions** of an angle assign values that quantify the relative orientation of the two rays as a function of the angle's measure.

The six trigonometric functions are:

Sine	sin
Cosine	cos
Tangent	tan
Cosecant	csc
Secant	sec
Cotangent	cot

There are two primary geometric methods for defining the trigonometric functions: the **right triangle method** and the **unit circle method**. In this chapter, the trigonometric functions are defined using the right triangle method. Later chapters extend these definitions using the unit circle method.

Right Triangle Definitions In the right triangle method, the trigonometric functions are defined for an acute angle θ in a right triangle as relationships among the lengths of the hypotenuse, the side adjacent to θ, and the side opposite θ:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}$$

Cosecant, secant, and cotangent are defined by the **reciprocal identities** as the reciprocals of the first three functions:

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adjacent}}{\text{opposite}} = \frac{\cos\theta}{\sin\theta}$$

For example, in a right triangle where the hypotenuse has length 5, the side adjacent to θ has length 4, and the side opposite θ has length 3, the trigonometric values of θ are:

$$\sin\theta = \frac{3}{5}$$

$$\cos\theta = \frac{4}{5}$$

$$\tan\theta = \frac{3}{4}$$

$$\csc\theta = \frac{5}{3}$$

$$\sec\theta = \frac{5}{4}$$

$$\cot\theta = \frac{4}{3}$$

2.2 Angles of Special Triangles

Algebraic and Transcendental Values Numbers are classified according to how they can be expressed. **Rational numbers** can be expressed as the ratio of two integers, such as $\frac{3}{4}$. Irrational numbers cannot be expressed as the ratio of two integers, such as π or $\sqrt{2}$.

Irrational numbers are further classified as **algebraic** or **transcendental**. Algebraic irrational numbers satisfy polynomial equations with integer coefficients and can therefore be expressed using algebraic expressions. For example, $\sqrt{2}$ is an algebraic irrational number since it satisfies the equation $x^2 - 2 = 0$. In contrast, numbers such as π and e do not satisfy any polynomial equation with integer coefficients and are transcendental.

The trigonometric values of most angles are transcendental and therefore cannot be expressed using algebraic expressions. Algebraic trigonometric values occur only for select **special angles**.

The special angles are **quadrantal angles** and the angles associated with **special triangles**. Quadrantal angles are treated in later chapters.

Special triangles are right triangles whose angles produce algebraic side-length ratios. There are two such triangles: the $45^\circ - 45^\circ - 90^\circ$ triangle and the $30^\circ - 60^\circ - 90^\circ$ triangle.

The $45^\circ - 45^\circ - 90^\circ$ Triangle

The $45^\circ - 45^\circ - 90^\circ$ triangle is an isosceles right triangle, in which the two legs have equal lengths and the two acute angles are equal. Since the two acute angles in any right triangle must sum to 90°, each acute angle in this triangle is 45°.

The **Pythagorean theorem** states that for a right triangle with legs of lengths a and b and hypotenuse of length c, the side lengths are related by:

$$a^2 + b^2 = c^2$$

For a $45^\circ - 45^\circ - 90^\circ$ triangle, let the length of each leg be 1. By the Pythagorean theorem, the length of the hypotenuse c is:

$$c^2 = 1^2 + 1^2 \implies c = \sqrt{2}$$

Thus, in a $45^\circ - 45^\circ - 90^\circ$ triangle, the hypotenuse is always $\sqrt{2}$ times the length of either leg.

Using these side lengths, the trigonometric values of a 45° angle are:

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$
$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$
$$\tan 45^\circ = 1$$
$$\csc 45^\circ = \sqrt{2}$$
$$\sec 45^\circ = \sqrt{2}$$
$$\cot 45^\circ = 1$$

The $30^\circ - 60^\circ - 90^\circ$ Triangle

The $30^\circ - 60^\circ - 90^\circ$ triangle is constructed from an equilateral triangle.

The **median** of a triangle is a line drawn from a vertex to the midpoint of the opposite side. In an equilateral triangle, the median is perpendicular to the opposite side and is also an **angle bisector**, meaning that it divides the vertex angle into two equal angles.

The vertices of a triangle are typically labeled A, B, and C, with the angles at the vertices denoted $\angle A$, $\angle B$, and $\angle C$. The sides are labeled by the opposite vertex: the side opposite A is labeled a, the side opposite B is labeled b, and the side opposite C is labeled c.

To construct a $30^\circ - 60^\circ - 90^\circ$ triangle, begin with an equilateral triangle and draw a median from vertex A to the midpoint of the opposite side a. Discard the half of the original triangle containing vertex C and side b. The remaining figure is a right triangle: the hypotenuse is the original side c, and the legs are the median and the remaining half of side a.

Let the median be denoted b_2, and let the remaining half of a be denoted a_2. Let the vertex where a_2 and b_2 meet be denoted C_2, and let the vertex of the new triangle at A be denoted A_2. The resulting right triangle has vertices A_2, B, and C_2, legs a_2 and b_2, and hypotenuse c.

Since b_2 is perpendicular to a_2, $\angle C_2 = 90^\circ$. The angle at B is unchanged, so $\angle B = 60^\circ$. Since the median bisects $\angle A$, $\angle A_2 = 30^\circ$.

Let the side lengths of the original equilateral triangle be 1. In the constructed right triangle, the hypotenuse c remains 1. The shorter leg a_2 is half of a, so it has length $\frac{1}{2}$. The length of the median leg b_2 is determined using the Pythagorean theorem:

$$b_2^2 + \left(\tfrac{1}{2}\right)^2 = 1^2$$
$$b_2^2 = \tfrac{3}{4}$$
$$b_2 = \tfrac{\sqrt{3}}{2}$$

Thus, in a $30^\circ - 60^\circ - 90^\circ$ triangle, the shorter leg is opposite the 30° angle, the hypotenuse is twice the length of the shorter leg, and the longer leg is opposite the 60° angle and is $\sqrt{3}$ times the length of the shorter leg.

Using these side lengths, the trigonometric values of 30° and 60° are:

$$\sin 30^\circ = \tfrac{1}{2}$$
$$\cos 30^\circ = \tfrac{\sqrt{3}}{2}$$
$$\tan 30^\circ = \tfrac{1}{\sqrt{3}}$$
$$\csc 30^\circ = 2$$

$\sec 30^\circ = \frac{2}{\sqrt{3}}$

$\cot 30^\circ = \sqrt{3}$

and:

$\sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos 60^\circ = \frac{1}{2}$

$\tan 60^\circ = \sqrt{3}$

$\csc 60^\circ = \frac{2}{\sqrt{3}}$

$\sec 60^\circ = 2$

$\cot 60^\circ = \frac{1}{\sqrt{3}}$

Note that the sine of 30° is the same as the cosine of 60°, and the cosine of 60° is the same as the sine of 30°. Since 30° and 60° are the two acute angles of the same right triangle, the side that is opposite one angle is adjacent to the other, so the sine of one angle is the same as the cosine of the other. This holds for any two **complementary angles**, meaning angles whose measures sum to 90°. This relationship will be developed further in later chapters.

2.3 Solving Right Triangles

Solving a triangle is the determination of the lengths of its three sides and the measures of its three angles.

In a right triangle, the hypotenuse is typically denoted as side c and the right angle opposite the hypotenuse is denoted as $\angle C$.

For a right triangle, the measure of $\angle C$ is given as 90°. Solving the triangle consists of determining the measures of the remaining angles $\angle A$ and $\angle B$ and the lengths of the legs a and b and the hypotenuse c.

If the measure of one acute angle and the length of one side are known, the remaining parts of the triangle can be determined using the trigonometric functions of the known angle.

For example, if the known side is the leg adjacent to the known angle, let the known angle be denoted $\angle A$, and let the length of the adjacent leg be denoted b. Since the acute angles of a right triangle are complementary, the remaining angle $\angle B$ is given by:

$\angle B = 90^\circ - \angle A$

The lengths of the remaining leg a and the hypotenuse c are found using the tangent and secant of the known angle together with the length of the known side.

$$\tan \angle A = \frac{a}{b}$$
$$a = b \tan \angle A$$
$$\sec \angle A = \frac{c}{b}$$
$$c = b \sec \angle A$$

Aside from special angles, trigonometric values are transcendental, so these calculations can be computed only to a given degree of precision.

Chapter 3

Trigonometric Functions on the Coordinate Plane

3.1 Extending Trigonometric Functions to the Coordinate Plane

Acute Angles in Standard Position The previous chapter defined the trigonometric functions as the relationships among the lengths of the sides of a right triangle.

For an angle θ in standard position, these functions can be realized for any point along the terminal ray by dropping a line from the point to the x-axis to construct a right triangle. Let the coordinates of the point be denoted (x, y) and let the distance along the terminal ray from the origin to (x, y) be denoted r. The hypotenuse of the constructed right triangle is the line segment along the terminal ray from the origin to (x, y) and has length r, the adjacent leg is the line segment along the x-axis from the origin to $(x, 0)$ and has length x, and the opposite leg is the dropped line from (x, y) to $(x, 0)$ and has length y.

The values of the trigonometric functions for the angle θ are:

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y}$$

$$\sec\theta = \frac{r}{x}$$

$$\cot\theta = \frac{x}{y}$$

This framing leads to a broader understanding of the trigonometric functions that extends beyond right triangles, as will be explained in the next section.

Horizontal and Vertical Extensions of an Angled Line When an angle is in standard position, a given length along the terminal ray is extended some fraction of that length horizontally, parallel to the initial ray, and some fraction of that length vertically, perpendicular to the initial ray. If the endpoint of this length is the point (x_t, y_t), then the horizontal extension has length x_t and the vertical extension has length y_t.

For a given angle, the ratios of the horizontal and vertical extensions to the length along the terminal ray are constant. Rotating the terminal ray to a different angle produces different horizontal and vertical extensions for the same length, and therefore different ratios.

The trigonometric functions represent these ratios and quantify the horizontal and vertical extensions of a line oriented at an angle. Sine is the vertical extension per unit of angled length, and cosine is the horizontal extension per unit of angled length.

While the right triangle provides a direct geometric realization of the trigonometric functions, the functions themselves are intrinsic to the orientation of an angled line.

Coterminal angles have different angle measures but the same trigonometric functions. The measure of an angle is defined as the amount of rotation from the initial ray to the terminal ray. In coterminal angles, the terminal rays share the same orientation relative to the initial ray, differing only by whole rotations.

3.2 Non-Acute Angles on the Coordinate Plane

Coordinate Plane Quadrants The coordinate plane is divided by the x and y axes into four **quadrants**. The upper right is QI, the upper left is QII, the lower left is $QIII$, and the lower right is QIV.

In QI, both x and y are positive. In QII, y is positive and x is negative. In $QIII$, both x and y are negative. In QIV, x is positive and y is negative.

Trigonometric Functions of Non-Acute Angles The right triangle definitions of the trigonometric functions apply only to acute angles, where the terminal ray lies in QI. However, the broader extension-based interpretation applies to all angles by considering a point (x, y) on the terminal ray at a distance of r from the origin.

The signs of the coordinates of a point along the terminal ray, and by extension the signs of the trigonometric functions, depend on which quadrant the terminal ray lies in. Note that while x and y may be positive or negative, r is always positive, since it represents the distance from the origin along the terminal ray.

For acute angles in QI, both x and y are positive, so both the horizontal and vertical extensions of the angled line are positive, resulting in a positive sine and cosine.

For obtuse angles in QII, x is negative while y is positive. The horizontal extension is parallel to the initial ray but oriented in the opposite direction, so it carries a negative sign. As a result, sine is positive and cosine is negative.

For reflex angles between 180° and 270°, where the terminal ray lies in $QIII$, both x and y are negative. Both the horizontal and vertical extensions are oriented in the negative directions, resulting in both sine and cosine being negative.

For reflex angles between 270° and 360°, where the terminal ray lies in QIV, x is positive and y is negative. The horizontal extension is positive while the vertical extension is negative, so cosine is positive and sine is negative.

Thus, sine is positive in QI and QII, and cosine is positive in QI and QIV. Tangent, defined as $\tan\theta = \frac{y}{x}$, is positive in QI and $QIII$. The reciprocal functions follow the same signs: cosecant is positive in QI and QII, secant is positive in QI and QIV, and cotangent is positive in QI and $QIII$.

Notes on the Coordinate Plane

It is important to clarify several aspects of the coordinate plane.

First, note that positive and negative here do not assert absolute qualities; rather, they distinguish opposite orientations along a given dimension.

Second, note that the directionality of the plane is not intrinsic but a matter of convention. This includes the choice that x is positive to the right and y is positive upward, as well as the choice that the initial direction is horizontal and the orthogonal direction is vertical.

The only intrinsic structure is that an angle consists of an initial direction and a second direction obtained by rotation that is orthogonal to it. Positive rotation is defined as rotation from the initial direction toward the chosen positive orthogonal direction.

Which axis represents the initial direction, and which orientations are taken as positive or negative, do not affect the underlying geometry or the trigonometric relationships themselves.

3.3 Reference Angles

The trigonometric functions have been expanded in this chapter to apply to and be directly calculated for any angle. Nevertheless, acute angles serve as **reference angles** from which the trigonometric values for non-acute angles can also be derived.

An angle's reference angle is the acute angle formed between the terminal ray of the given angle and the x-axis of the quadrant in which the terminal ray lies. For angles in $\boldsymbol{QI}$ and $\boldsymbol{QIV}$, this axis is the positive x-axis. For angles in $\boldsymbol{QII}$ and $\boldsymbol{QIII}$, this axis is the negative x-axis.

This rotation is never greater than 90°, so all reference angles are acute and lie in $\boldsymbol{QI}$.

The magnitudes of an angle's trigonometric functions are equal to those of its reference angle; however, the signs of these values depend on the quadrant in which the terminal ray lies: $\sin$ and $\csc$ are positive in $\boldsymbol{QI}$ and $\boldsymbol{QII}$ where y is positive; $\cos$ and $\sec$ are positive in $\boldsymbol{QI}$ and $\boldsymbol{QIV}$ where x is positive; and $\tan$ and $\cot$ are positive in $\boldsymbol{QI}$ and $\boldsymbol{QIII}$ where x and y have the same sign.

Thus, the trigonometric values for any non-acute angle can be obtained by determining its reference angle and assigning signs according to the angle's quadrant.

For example, to find the trigonometric values of 690°:

First, determine a coterminal angle between 0° and 360°:

$$690^\circ - 360^\circ = 330^\circ$$

The terminal ray of 330° lies in $\boldsymbol{QIV}$. Its reference angle is the acute angle between the terminal ray and the positive x-axis:

$$360^\circ - 330^\circ = 30^\circ$$

The trigonometric values of 30° have already been determined in the previous chapter.

In $\boldsymbol{QIV}$, sine, cosecant, tangent, and cotangent are negative, while cosine and secant are positive.

Applying the appropriate signs gives:

$$\sin 330^\circ = -\frac{1}{2}$$

$$\cos 330^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 330^\circ = -\frac{1}{\sqrt{3}}$$

$$\csc 330^\circ = -2$$

$$\sec 330^\circ = \frac{2}{\sqrt{3}}$$

$$\cot 330^\circ = -\sqrt{3}$$

Chapter 4

Unit Circle Method

4.1 Unit Circle Definitions

Normalization of the Terminal Ray Length Earlier in this chapter, the trigonometric functions were interpreted as the ratios relating a length r measured along the terminal ray of an angle to the corresponding x- and y-coordinates of the point at the end of that length, with sine given by $\frac{y}{r}$ and cosine by $\frac{x}{r}$.

If the length r is fixed to be one unit, then the x- and y-coordinates of the point directly give the values of cosine and sine, without the need for scaling by r.

Thus, for any angle in standard position, the coordinates of the point on its terminal ray that lies one unit from the origin determine the sine and cosine of the angle.

This normalization establishes a standard geometric framework for defining the trigonometric functions, known as the **unit circle method**.

The Unit Circle

The **unit circle** is the set of all points in the coordinate plane that are a distance of 1 unit from the origin. These points form a circle of radius 1 and satisfy the equation

$$x^2 + y^2 = 1$$

Let t denote the length of a path measured counterclockwise along the unit circle starting from the point $(1, 0)$. The point (x, y) on the unit circle reached after traversing a distance t is called the **terminal point** determined by t.

Since the circumference of a circle of radius 1 is 2π, the terminal point corresponding to $t = 2\pi$ is $(1, 0)$, which lies on the positive x-axis. The points on the remaining three coordinate axes are:

t	**Terminal Point**	**Axis**
$t = \frac{\pi}{2}$	$(0, 1)$	Positive y-axis
$t = \pi$	$(-1, 0)$	Negative x-axis
$t = \frac{3\pi}{2}$	$(0, -1)$	Negative y-axis

The terminal point determined by t corresponds to the point on the terminal ray of an angle with radian measure t that lies one unit from the origin.

From this point onward, angles will generally be measured in radians rather than degrees. The unit rad will be omitted, and all unspecified angle measures should be assumed to be in radians.

The **unit circle method** defines the trigonometric functions as functions of the number t using the coordinates (x, y) of the terminal point determined by t:

$$\sin t = y$$

$$\cos t = x$$

The remaining four trigonometric functions follow as:

$$\tan t = \frac{\sin t}{\cos t} = \frac{y}{x}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{y}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{x}$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t} = \frac{x}{y}$$

This method defines the trigonometric functions in terms of arc length along the unit circle rather than directly in terms of angle measure. Within the context of angular geometry, the radian measure of an angle is defined by the length of the corresponding arc, so values defined by arc length apply directly to radian angle measure, and the change in underlying definition does not affect the resulting relationships. However, the fact that the functions are defined in this way will become significant in later chapters and will be explored further there.

Quadrantal Angles

Angles whose terminal ray lies on one of the four coordinate axes are called **quadrantal angles**. The trigonometric functions are undefined for quadrantal angles whenever their definitions involve division by 0.

For tangent and secant, which have the x-coordinate in their denominator, this occurs at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, where the terminal ray lies on the positive and negative y-axis and the terminal point is $(0, 1)$ and $(0, -1)$, respectively.

For cotangent and cosecant, which have the y-coordinate in their denominator, this occurs at 0 and π, where the terminal ray lies on the positive and negative x-axis and the terminal point is $(1, 0)$ and $(-1, 0)$, respectively.

The trigonometric values at the four quadrantal angles are:

Function	$\theta = 0$	$\theta = \frac{\pi}{2}$	$\theta = \pi$	$\theta = \frac{3\pi}{2}$
$\sin\theta$	0	1	0	-1
$\cos\theta$	1	0	-1	0
$\csc\theta$	undefined	1	undefined	-1
$\sec\theta$	1	undefined	-1	undefined
$\tan\theta$	0	undefined	0	undefined
$\cot\theta$	undefined	0	undefined	0

Pythagorean Identities

For any point on the unit circle corresponding to an angle θ, a right triangle can be constructed with hypotenuse of length 1 and legs of lengths $\sin\theta$ and $\cos\theta$. By the Pythagorean Theorem, the sum of the squares of the lengths of the legs of a right triangle equals the square of the length of the hypotenuse. Therefore,

$$\sin^2\theta + \cos^2\theta = 1$$

for any angle θ. This identity is called the **Pythagorean identity**.

For trigonometric functions, expressions such as $\sin^2\theta$ denote $(\sin\theta)^2$, rather than function composition.

Two additional identities are obtained by dividing the Pythagorean identity by $\sin^2\theta$ and by $\cos^2\theta$.

Dividing by $\sin^2\theta$ gives:

$$\begin{aligned}\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta}\\ 1 + \cot^2\theta &= \csc^2\theta\end{aligned}$$

Dividing by $\cos^2\theta$ gives:

$$\begin{aligned}\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta}\\ \tan^2\theta + 1 &= \sec^2\theta\end{aligned}$$

Together, the three Pythagorean identities are:

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

Reference Numbers As established in an earlier chapter, the magnitudes of the trigonometric function values for a given angle are equal to those of its reference angle and differ only in sign.

Reference angles were previously defined as the acute angles formed between an angle's terminal ray and the x-axis of the quadrant in which the terminal ray lies.

Under the unit circle method, trigonometric functions are defined for numbers corresponding to a distance t measured along the unit circle rather than for angles themselves. Accordingly, reference angles are replaced by **reference numbers**, which are defined as the distance measured along the unit circle from the terminal point corresponding to $t = \theta$ to the x-axis of its quadrant.

As noted earlier, this shift from angles to numbers does not affect angular geometry. The purpose of introducing reference numbers in this section is to update the concept of reference angles to align with the unit circle definition. For practical calculation of reference angles and the derivation of trigonometric values, the procedures remain unchanged.

4.2 Special Angles

Special Angles on the Unit Circle In an earlier chapter, the trigonometric values for special angles were derived from special right triangles. These values can be applied directly to the unit circle:

Angle (°)	**Angle** (rad)	sin	cos	**Unit circle coordinates**
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

These angles serve as reference angles for the corresponding special angles in the other three quadrants.

For QII:

Angle	Reference angle	Unit circle coordinates
$\frac{2\pi}{3}$	$\frac{\pi}{3}$	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{3\pi}{4}$	$\frac{\pi}{4}$	$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{5\pi}{6}$	$\frac{\pi}{6}$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

For $QIII$:

Angle	Reference angle	Unit circle coordinates
$\frac{4\pi}{3}$	$\frac{\pi}{3}$	$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\frac{5\pi}{4}$	$\frac{\pi}{4}$	$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\frac{7\pi}{6}$	$\frac{\pi}{6}$	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

For QIV:

Angle	Reference angle	Unit circle coordinates
$\frac{5\pi}{3}$	$\frac{\pi}{3}$	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\frac{7\pi}{4}$	$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\frac{11\pi}{6}$	$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

Procedure for Determining Trigonometric Values

The following summarizes the rules introduced thus far for determining the trigonometric values associated with a given angle θ:

If θ does not lie in the interval $[0, 2\pi)$, it can be replaced by its principal coterminal angle, which shares the same coordinates.

If θ is a quadrantal angle, the coordinates of its terminal point are:

Angle	Coordinates
$\theta = 0$	$(1, 0)$
$\theta = \frac{\pi}{2}$	$(0, 1)$
$\theta = \pi$	$(-1, 0)$
$\theta = \frac{3\pi}{2}$	$(0, -1)$

For non-quadrantal angles, the signs of the coordinates of the terminal point are determined by the quadrant in which the terminal point lies, which is in turn determined by the interval containing the principal coterminal angle.

Interval	Quadrant	Sign of x	Sign of y
$(0, \frac{\pi}{2})$	QI	$+$	$+$
$(\frac{\pi}{2}, \pi)$	QII	$-$	$+$
$(\pi, \frac{3\pi}{2})$	$QIII$	$-$	$-$
$(\frac{3\pi}{2}, 2\pi)$	QIV	$+$	$-$

For angles not in $\boldsymbol{QI}$, the magnitudes of the coordinates are equal to those of a corresponding reference angle in $\boldsymbol{QI}$ and differ only in sign. The reference angle is given by:

Quadrant	**Reference angle**
$\boldsymbol{QII}$	$\pi - \theta$
$\boldsymbol{QIII}$	$\lvert\pi - \theta\rvert$
$\boldsymbol{QIV}$	$2\pi - \theta$

Chapter 5

Geometric Interpretations from the Unit Circle

For some, it is helpful to visualize the trigonometric functions directly from the geometry of the unit circle. The following descriptions are intended to aid in this regard.

5.1 Trigonometric Functions

Sine and Cosine For a point on the unit circle, $\sin\theta$ equals the y-coordinate and $\cos\theta$ equals the x-coordinate. Accordingly, sine represents vertical position and cosine represents horizontal position.

In the coordinate plane, vertical position is positive above the x-axis and negative below it, while horizontal position is positive to the right of the y-axis and negative to the left. Consequently, on the unit circle, sine is positive in the upper half and negative in the lower half, while cosine is positive in the right half and negative in the left half.

Consider a point moving continuously around the unit circle as the angle θ varies, and examine its vertical and horizontal components separately.

For the vertical component, the point lies at $y = 0$ when $\theta = 0$ and increases to $y = 1$ as θ increases to $\frac{\pi}{2}$. As θ continues to increase, the vertical position decreases to $y = 0$ at π, then continues decreasing to $y = -1$ at $\frac{3\pi}{2}$. Beyond this point, the vertical position increases again, returning to $y = 0$ at 2π. Thus, over one complete cycle, $\sin\theta$ varies continuously from 0 to 1, back to 0, then to -1, and finally returns to 0.

The horizontal component follows a similar pattern, but shifted in phase. The point lies at $x = 1$ when $\theta = 0$ and decreases continuously, passing through $x = 0$ at $\frac{\pi}{2}$ and reaching $x = -1$ at π. As θ increases further, the horizontal position increases again, passing through $x = 0$ at $\frac{3\pi}{2}$ and returning to $x = 1$ at 2π. Thus, over one complete cycle, $\cos\theta$ varies from 1 to 0, then to -1, then increases to 0, and finally returns to 1.

In this way, both the signs and approximate magnitudes of $\sin\theta$ and $\cos\theta$ may be inferred directly from the position of the terminal point on the unit circle.

Tangent Unlike sine and cosine, which describe horizontal and vertical components, the tangent of an angle describes the slope of its terminal ray. The sign of tangent is therefore determined by the orientation of this ray: where the ray has positive slope, as in QI and $QIII$, $\tan\theta$ is positive; where the ray has negative slope, as in QII and QIV, $\tan\theta$ is negative.

Reciprocal Trigonometric Functions The reciprocal trigonometric functions may be visualized as geometric reflections of their corresponding primary trigonometric functions across specific lines.

Cosecant Cosecant is the reciprocal of sine. On the unit circle, $\sin\theta$ represents vertical position within the range $[-1, 1]$. The value of $\csc\theta$ may be visualized as this value reflected nonlinearly over the bounding lines $y = 1$ and $y = -1$.

At the bounds of the interval, where $\sin\theta = 1$, $\csc\theta = 1$, and where $\sin\theta = -1$, $\csc\theta = -1$. As $\sin\theta$ moves downward from 1 toward 0, $\csc\theta$ moves upward from 1; the further $\sin\theta$ moves from 1, the more rapidly $\csc\theta$ increases. As $\sin\theta$ approaches 0, $\csc\theta$ approaches infinity and is undefined when $\sin\theta = 0$.

The same applies for negative values. As $\sin\theta$ moves upward from -1 toward 0, the reflected value moves downward from -1 toward $-\infty$, with the rate increasing as $\sin\theta$ approaches 0.

Secant Secant is the reciprocal of cosine and may be treated in the same manner as cosecant and sine.

On the unit circle, $\cos\theta$ represents horizontal position within the range $[-1, 1]$, and $\sec\theta$ may be visualized as this value reflected over the bounding lines $x = 1$ and $x = -1$. At the bounds, where $\cos\theta = 1$ or $\cos\theta = -1$, $\sec\theta$ equals $\cos\theta$. As $\cos\theta$ decreases in magnitude toward 0, $\sec\theta$ increases in magnitude nonlinearly toward infinity.

Cotangent Cotangent is the reciprocal of tangent. With tangent viewed as the slope of the terminal ray, cotangent may be visualized as the slope of the ray formed by reflecting the terminal ray within its quadrant over the central line. This swaps the horizontal and vertical components of the ray, yielding the reciprocal slope.

5.2 Special Angles

Angle Measures of the Special Angles Each quadrant of the unit circle contains three special angles: one at the center of the quadrant, one closer to the x-axis, and one closer to the y-axis.

The measures of these angles may be viewed relative to a half rotation of the unit circle. Each half rotation is bounded by quadrantal angles on the x-axis, which occur at integer multiples of π, and spans an angular measure of π, including either QI and QII or $QIII$ and QIV.

Within each such half rotation, the quadrantal angle on the y-axis lies halfway through the interval, and therefore occurs at an angular measure of $\frac{\pi}{2}$ from the starting point of the half rotation.

The remaining special angles occur at fixed fractional positions within the half rotation. The angle at the center of each quadrant lies halfway through the quadrant, corresponding to one quarter and three quarters of the half rotation, and therefore occurs at angular measures of $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ from the starting point.

The special angle closer to the y-axis lies at one third of the half rotation, occurring at angular measures of $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ from the starting point.

The special angle closer to the x-axis lies at one sixth of the half rotation, occurring at angular measures of $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ from the starting point.

Sine and Cosine of the Special Angles For the sine and cosine values of the special angles, there are only three possible magnitudes: $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, and $\frac{\sqrt{3}}{2}$. These correspond to the square roots of the integers 1, 2, and 3, respectively, each divided by 2.

For the central special angles in each quadrant, the horizontal and vertical components have equal magnitude. Consequently, both sine and cosine take the intermediate value $\frac{\sqrt{2}}{2}$.

For the special angles nearer the sides of each quadrant, the horizontal and vertical components have unequal magnitudes. In all such cases, the smaller component has magnitude $\frac{1}{2}$ and the larger has magnitude $\frac{\sqrt{3}}{2}$.

Tangent of the Special Angles For the special angles, the magnitude of tangent takes one of three values: 1, $\sqrt{3}$, or $\frac{\sqrt{3}}{3}$.

For the central special angle in each quadrant, the horizontal and vertical components have equal magnitude, so the slope has magnitude 1.

For the special angles toward the sides of each quadrant, the angle closer to the y-axis is steeper and has slope $\sqrt{3}$, while the angle closer to the x-axis is flatter and has slope $\frac{\sqrt{3}}{3}$.

Chapter 6

Graphs of the Trigonometric Functions

6.1 Trigonometric Functions Over a Range of Inputs

Previous chapters examined the trigonometric functions at individual input values θ. In this chapter, the focus shifts to the full range of values these functions assume as θ varies continuously, by studying their graphs.

For a trigonometric function f, the basic behavior of the function is represented by the graph of the equation

$$y = f(x)$$

In addition to the basic graphs, transformations of these functions are studied. For a trigonometric function f, the general transformed form is

$$y = a\ f\,(b\,(x + c)) + d$$

When the sine or cosine function appears in this form, the resulting graph is called a **sinusoidal function**.

This chapter also examines the **parity** of the trigonometric functions. The parity of a function describes whether it is **odd** or **even**. A function is odd if changing the sign of the input changes the sign of the output, and even if it does not.

For an odd function f,

$$f(-x) = -f(x)$$

For an even function f,

$$f(-x) = f(x)$$

Periodic Functions

A **periodic function** is a function whose outputs follow a fixed pattern of values that repeats at regular intervals of the input. The length of this repeating input interval is called the **period** of the function.

For a function f with period P, the following holds for every value x in the domain of f:

$$f(x) = f(x + P)$$

The graph of a periodic function consists of a single characteristic shape that repeats at regular intervals along the horizontal axis. Each such interval represents one complete period of the function. The entire graph of a periodic function is determined by its behavior over any single period.

Periodicity of the Trigonometric Functions The trigonometric functions are periodic; their values repeat with each complete rotation around the unit circle. For a trigonometric function f,

$$f(x+2\pi) = f(x)$$

The sine, cosine, cosecant, and secant functions have period 2π, since their values repeat after one full rotation. In contrast, the tangent and cotangent functions repeat their values after a half-rotation, and therefore have period π. This distinction is examined later in this chapter.

Because the trigonometric functions are periodic, their graphs are completely determined by their behavior over a single period.

6.2 Graph of the Sine Function

Description of the Function

The function $\sin\theta$ represents the vertical coordinate of the terminal point located a distance θ along the unit circle.

- At $\theta = 0$, the terminal point lies on the x-axis at $y = 0$.
- Over the interval $[0, \frac{\pi}{2}]$, the terminal point traverses the unit circle in QI, rising from $y = 0$ to $y = 1$.
- Over the interval $[\frac{\pi}{2}, \pi]$, the point traverses QII, falling from $y = 1$ back to $y = 0$.
- Over the interval $[\pi, \frac{3\pi}{2}]$, the point traverses $QIII$, falling from $y = 0$ to $y = -1$.
- Over the interval $[\frac{3\pi}{2}, 2\pi]$, the point traverses QIV, rising from $y = -1$ back to $y = 0$.
- At $\theta = 2\pi$, the terminal point again lies on the x-axis at $y = 0$, and the cycle repeats.

The sine function is an odd function. Starting from the initial point of the unit circle, $(1,0)$, positive and negative rotations produce vertical displacements of equal magnitude and opposite sign. Thus,

$$\sin(-x) = -\sin x.$$

Graph of the Basic Function

The graph of the function $y = \sin x$ represents the vertical coordinate of the terminal point as a function of the input.

The period of the function is 2π.

By convention, a single reference period is represented beginning at 0, spanning the interval $[0, 2\pi]$. Over this interval, the graph consists of four equal segments, each of width $\frac{\pi}{2}$.

- At $x = 0$, $\sin x = 0$.
- On $[0, \frac{\pi}{2}]$, $\sin x$ increases from 0 to 1.
- On $[\frac{\pi}{2}, \pi]$, $\sin x$ decreases from 1 to 0.
- On $[\pi, \frac{3\pi}{2}]$, $\sin x$ decreases from 0 to -1.
- On $[\frac{3\pi}{2}, 2\pi]$, $\sin x$ increases from -1 to 0.
- At $x = 2\pi$, $\sin x = 0$, and the pattern repeats.

The overall shape of the graph may be geometrically described as resembling the upper half of a circle followed by the lower half of a circle.

This pattern repeats for every integer n, beginning at $2\pi n$ and spanning the subsequent 2π interval $[2\pi n, 2\pi(n+1)]$.

The values 0, 1, and -1 correspond to the vertical midline, upper extremum, and lower extremum of the graph. The distance from the midline to either extremum is the **amplitude** of the function. For the basic sine function, the amplitude is 1.

Transformations

The graph of $y = \sin x$ may be transformed in the following ways.

For $y = \sin(x+c)$, the graph is shifted horizontally by $-c$. The pattern repeats beginning at $-c+2\pi n$ instead of $2\pi n$ and spans the interval $[-c+2\pi n, -c+2\pi(n+1)]$. The horizontal shift of a periodic function is called the **phase shift**.

For $y = \sin x + d$, the graph is shifted vertically by d. The vertical midline is located at $y = d$ instead of $y = 0$.

For $y = \sin(bx)$, the graph is compressed horizontally toward the y-axis by a factor of b. The period becomes $\frac{2\pi}{b}$. The pattern repeats beginning at $\frac{2\pi}{b}n$ and spans the subsequent $\frac{2\pi}{b}$ interval $\left[\frac{2\pi}{b}n, \frac{2\pi}{b}(n+1)\right]$.

For $y = a(\sin x)$, the graph is stretched vertically from the midline by a factor of $|a|$. The amplitude is $|a|$, instead of 1, and the upper and lower extrema are at a distance $|a|$ from the midline.

For $y = \sin(-x)$, the graph is reflected across the x-axis, since the sine function is odd.

For $y = -\sin x$, the graph is reflected across the x-axis.

Graph of the General Function

For the general sine function:

$$y = a\sin(b(x+c)) + d$$

the following properties apply:

- The period is $\frac{2\pi}{b}$.
- The phase shift is $-c$.
- Each period begins at $-c + \frac{2\pi}{b}n$ and spans the subsequent $\frac{2\pi}{b}$ interval $\left[-c + \frac{2\pi}{b}n, -c + \frac{2\pi}{b}(n+1)\right]$.
- The vertical midline is $y = d$.
- The amplitude is $|a|$.
- The upper and lower extrema are $d + |a|$ and $d - |a|$.
- A single period consists of four segments, each of width $\frac{\pi}{2b}$, over which the graph:
  - rises from the midline to the upper extremum,
  - falls from the upper extremum to the midline,
  - falls from the midline to the lower extremum,
  - rises from the lower extremum back to the midline.
- If a or b is negative, the graph is reflected across the x-axis.
- The domain of the function is all real numbers.
- The range of the function is $[\,d - |a|, d + |a|\,]$.

6.3 Graph of the Cosine Function

Description of the Function

The function $\cos\theta$ represents the horizontal coordinate of the terminal point located a distance θ along the unit circle.

- At $\theta = 0$, the terminal point lies on the x-axis at $x = 1$.
- Over the interval $[0, \frac{\pi}{2}]$, the terminal point traverses the unit circle in QI, moving leftward from $x = 1$ to $x = 0$.
- Over the interval $[\frac{\pi}{2}, \pi]$, the point traverses QII, moving leftward from $x = 0$ to $x = -1$.
- Over the interval $[\pi, \frac{3\pi}{2}]$, the point traverses $QIII$, moving rightward from $x = -1$ to $x = 0$.
- Over the interval $[\frac{3\pi}{2}, 2\pi]$, the point traverses QIV, moving rightward from $x = 0$ back to $x = 1$.
- At $\theta = 2\pi$, the terminal point again lies on the x-axis at $x = 1$, and the cycle repeats.

The cosine function is an even function. Starting from the initial point of the unit circle, $(1, 0)$, positive and negative rotations produce horizontal displacements of equal magnitude and identical sign. Thus,

$$\cos(-x) = \cos x.$$

Graph of the Basic Function

The graph of the function $y = \cos x$ represents the horizontal coordinate of the terminal point as a function of the input.

The period of the function is 2π.

By convention, a single reference period is represented beginning at 0, spanning the interval $[0, 2\pi]$. Over this interval, the graph consists of four equal segments, each of width $\frac{\pi}{2}$.

- At $x = 0$, $\cos x = 1$.
- On $[0, \frac{\pi}{2}]$, $\cos x$ decreases from 1 to 0.
- On $[\frac{\pi}{2}, \pi]$, $\cos x$ decreases from 0 to -1.
- On $[\pi, \frac{3\pi}{2}]$, $\cos x$ increases from -1 to 0.
- On $[\frac{3\pi}{2}, 2\pi]$, $\cos x$ increases from 0 to 1.
- At $x = 2\pi$, $\cos x = 1$, and the pattern repeats.

The overall shape of the graph may be geometrically described as resembling the upper right quarter of a circle, followed by the lower half of a circle, and then the upper left quarter of a circle.

This pattern repeats for every integer n, beginning at $2\pi n$ and spanning the subsequent 2π interval $[2\pi n, 2\pi(n+1)]$.

The vertical midline is $y = 0$, and the amplitude is 1.

Graph of the General Function

The graph of cosine is transformed in much the same way as that of sine, except that cosine is an even function.

For the general cosine function:

$y = a\cos(b(x + c)) + d$

the following properties apply:

- The period is $\frac{2\pi}{b}$.
- The phase shift is $-c$.
- Each period begins at $-c + \frac{2\pi}{b}n$ and spans the subsequent $\frac{2\pi}{b}$ interval $\left[-c + \frac{2\pi}{b}n, -c + \frac{2\pi}{b}(n+1)\right]$.
- The vertical midline is $y = d$.
- The amplitude is $|a|$.
- The upper and lower extrema are $d + |a|$ and $d - |a|$.
- A single period consists of four segments, each of width $\frac{\pi}{2b}$, over which the graph:
  - decreases from the upper extremum to the midline,
  - decreases from the midline to the lower extremum,
  - increases from the lower extremum to the midline,
  - increases from the midline to the upper extremum.
- If b is negative, the graph is unaffected, since cosine is an even function.
- If a is negative, the graph is reflected across the x-axis.
- The domain of the function is all real numbers.
- The range of the function is $[\,d - |a|, d + |a|\,]$.

6.4 Graph of the Cosecant Function

Description of the Function

The function $\csc\theta$ is the reciprocal of $\sin\theta$.

- Where $\sin\theta = \pm 1$, $\csc\theta = \pm 1$.
- Where $\sin\theta = 0$, $\csc\theta$ is undefined.
- As $\sin\theta$ decreases in magnitude from ± 1 toward 0, $\csc\theta$ increases in magnitude from ± 1 toward $\pm\infty$.
- As $\sin\theta$ increases in magnitude from 0 toward ± 1, $\csc\theta$ decreases in magnitude from $\pm\infty$ toward ± 1.

Graph of the Basic Function

The graph of $y = \csc x$ may be viewed as the graph of $y = \sin x$ transformed reciprocally about the amplitude lines at $y = \pm 1$. As $\sin x$ moves from ± 1 toward 0, the corresponding values of $\csc x$ diverge from ± 1 toward $\pm\infty$ at an increasingly rapid rate, so that equal changes in $\sin x$ produce progressively larger changes in $\csc x$. In this way, the region of the sine graph between the midline and the amplitude lines is stretched outward in the cosecant graph beyond the amplitude lines toward $\pm\infty$.

The graph of $y = \csc x$ is described over the same reference interval as that of $y = \sin x$, $[0, 2\pi]$:

- At $x = 0$, $\sin x = 0$, so $\csc x$ is undefined and there is a vertical asymptote.
- On $[0, \frac{\pi}{2}]$, as $\sin x$ increases from 0 to 1, $\csc x$ decreases from ∞ to 1.
- On $[\frac{\pi}{2}, \pi]$, as $\sin x$ decreases from 1 to 0, $\csc x$ increases from 1 to ∞.
- At $x = \pi$, $\sin x = 0$, so $\csc x$ is undefined and there is a vertical asymptote.
- On $[\pi, \frac{3\pi}{2}]$, as $\sin x$ decreases from 0 to -1, $\csc x$ increases from $-\infty$ to -1.
- On $[\frac{3\pi}{2}, 2\pi]$, as $\sin x$ increases from -1 to 0, $\csc x$ decreases from -1 to $-\infty$.
- At $x = 2\pi$, $\sin x = 0$, so $\csc x$ is undefined, there is a vertical asymptote, and the period repeats.

The overall shape of the graph may be geometrically described as being defined by three evenly spaced vertical asymptotes, two of which frame the period on the left and right and a third at the center dividing it into two halves. Over the left half of the interval, the bottom half of a circle is stretched downward from the top of the graph toward the upper bounding line at $y = 1$, while over the right half, the top half of a circle is stretched upward from the bottom of the graph toward the lower bounding line at $y = -1$.

Unlike sine, whose domain is all real numbers, the domain of cosecant excludes values for which $\sin x = 0$. For the basic function $y = \csc x$, this occurs at integer multiples of π:

$$\mathrm{Dom}(\csc x) = \{\, x \in \mathbb{R} \mid x \neq k\pi,\ k \in \mathbb{Z} \,\}$$

The range of sine is $[-1, 1]$. The range of cosecant is $(-\infty, -1] \cup [1, \infty)$.

Graph of the General Function

The graph of cosecant is transformed in much the same way as that of sine. Since cosecant is the reciprocal of sine, it has the same parity and is an odd function.

For the general cosecant function

$$y = a\ \csc(b(x + c)) + d$$

the following properties apply:

- The period is $\frac{2\pi}{b}$.
- The phase shift is $-c$.
- Each period begins at $-c + \frac{2\pi}{b}n$ and spans the subsequent $\frac{2\pi}{b}$ interval $\left[-c + \frac{2\pi}{b}n,\ -c + \frac{2\pi}{b}(n+1)\right]$.
- The vertical midline is $y = d$.
- The vertical distance from the midline to each bounding line is $|a|$.
- The upper bounding line is $y = d + |a|$, and the lower bounding line is $y = d - |a|$.
- Vertical asymptotes occur at the start, center, and end of each period.

- A single period consists of four segments, each of width $\frac{\pi}{2b}$, over which the graph:
  - decreases from ∞ to the upper bounding line,
  - increases from the upper bounding line to ∞,
  - increases from $-\infty$ to the lower bounding line,
  - decreases from the lower bounding line to $-\infty$.
- If a or b is negative, the graph is reflected across the x-axis.
- The domain is all real numbers except at the vertical asymptotes:
  $$\mathrm{Dom}\big(a\ \csc(b(x+c))+d\big) = \{\ x \in \mathbb{R} \mid x \neq -c + \tfrac{k\pi}{b},\ k \in \mathbb{Z}\ \}$$
- The range is $(-\infty,\ d-|a|] \cup [d+|a|,\ \infty)$.

6.5 Graph of the Secant Function

Description of the Function

The function $\sec\theta$ is the reciprocal of $\cos\theta$. Its values are related to those of cosine in the same way that the values of cosecant are related to those of sine through reciprocal stretching about the bounding lines.

Graph of the Basic Function

The graph of $y = \sec x$ is described over the reference interval $[0, 2\pi]$:

- At $x = 0$, where $\cos x = 1$, $\sec x = 1$.
- On $[0, \frac{\pi}{2}]$, as $\cos x$ decreases from 1 to 0, $\sec x$ increases from 1 to ∞.
- At $x = \frac{\pi}{2}$, where $\cos x = 0$, $\sec x$ is undefined and there is a vertical asymptote.
- On $[\frac{\pi}{2}, \pi]$, as $\cos x$ decreases from 0 to -1, $\sec x$ increases from $-\infty$ to -1.
- On $[\pi, \frac{3\pi}{2}]$, as $\cos x$ increases from -1 to 0, $\sec x$ decreases from -1 to $-\infty$.
- At $x = \frac{3\pi}{2}$, where $\cos x = 0$, $\sec x$ is undefined and there is a vertical asymptote.
- On $[\frac{3\pi}{2}, 2\pi]$, as $\cos x$ increases from 0 to 1, $\sec x$ decreases from ∞ to 1.
- At $x = 2\pi$, where $\cos x = 1$, $\sec x = 1$, and the period repeats.

The overall shape of the graph may be geometrically described as being defined by two evenly spaced vertical asymptotes that divide the period into three regions. In the left region, the bottom right quarter of a circle is stretched downward from the top of the graph toward the upper bounding line at $y = 1$; in the central region, the top half of a circle is stretched upward from the bottom of the graph toward the lower bounding line at $y = -1$; and in the right region, the bottom left quarter of a circle is stretched downward from the top of the graph toward the upper bounding line at $y = 1$.

The domain of secant excludes values for which $\cos x = 0$. For the basic function $y = \sec x$, this occurs at $\frac{\pi}{2} + k\pi$ for any integer k:

$$\mathrm{Dom}(\sec x) = \{\ x \in \mathbb{R} \mid x \neq \tfrac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\ \}$$

The range of secant is $(-\infty, -1] \cup [1, \infty)$.

Graph of the General Function

Like cosine, secant is an even function.

For the general secant function

$$y = a \sec(b(x + c)) + d$$

the following properties apply:

- The period is $\frac{2\pi}{b}$.
- The phase shift is $-c$.
- Each period begins at $-c + \frac{2\pi}{b}n$ and spans the subsequent $\frac{2\pi}{b}$ interval $\left[-c + \frac{2\pi}{b}n,\ -c + \frac{2\pi}{b}(n+1)\right]$.
- The vertical midline is $y = d$.
- The distance from the midline to each bounding line is $|a|$.
- The upper and lower bounding lines are $y = d + |a|$ and $y = d - |a|$, respectively.
- Vertical asymptotes occur at the one-quarter and three-quarter points of each period.
- A single period consists of four segments, each of width $\frac{\pi}{2b}$, over which the graph:
  - increases from the upper bounding line toward ∞,
  - increases from $-\infty$ to the lower bounding line,
  - decreases from the lower bounding line toward $-\infty$,
  - decreases from ∞ to the upper bounding line.
- If b is negative, the graph is unaffected.
- If a is negative, the graph is reflected across the x-axis.
- The domain is all real numbers except at the vertical asymptotes:
  $$\mathrm{Dom}\big(a \sec(b(x + c)) + d\big) = \{\, x \in \mathbb{R} \mid x \neq -c + \tfrac{\pi}{2b} + \tfrac{k\pi}{b},\ k \in \mathbb{Z} \,\}$$
- The range is $(-\infty,\ d - |a|] \cup [d + |a|,\ \infty)$.

6.6 Graph of the Tangent Function

Description of the Function

The function $\tan\theta$ is defined as

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Where $\sin\theta = 0$, $\tan\theta = 0$. Where $\cos\theta = 0$, $\tan\theta$ is undefined.

At the quadrantal angles:

- At $\theta = 0$, $\sin\theta = 0$ and $\cos\theta = 1$, so $\tan\theta = 0$.
- At $\theta = \frac{\pi}{2}$, $\sin\theta = 1$ and $\cos\theta = 0$, so $\tan\theta$ is undefined.
- At $\theta = \pi$, $\sin\theta = 0$ and $\cos\theta = -1$, so $\tan\theta = 0$.
- At $\theta = \frac{3\pi}{2}$, $\sin\theta = -1$ and $\cos\theta = 0$, so $\tan\theta$ is undefined.

Period Size Although the sine and cosine functions have period 2π, the tangent function has period π.

The sequences of values of sine and cosine over QI have the same magnitudes as those over $QIII$, but opposite signs: both functions are positive in QI and negative in $QIII$. While the values of sine and cosine themselves therefore differ between these quadrants, their quotient, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, is positive in both and thus assumes identical values in the two quadrants.

The same occurs over QII and QIV. The sequences of values of sine and cosine again have the same magnitudes but opposite signs: sine is positive in QII and negative in QIV, while cosine is negative in QII and positive in QIV. Here too, although the values of sine and cosine themselves differ, their quotient is negative in both quadrants and assumes the same values.

Thus, although sine and cosine do not repeat their values until a full rotation is completed, giving them a period of 2π, the values of tangent repeat after a half rotation, so the period of the tangent function is π rather than 2π.

Reference Interval For sine and cosine, the reference interval is conventionally given beginning at $\theta = 0$.

In contrast, the tangent function has a vertical asymptote at $\theta = \frac{\pi}{2}$. If its reference interval were chosen beginning at 0, the period would be split by a vertical asymptote. For this reason, the reference interval for tangent is taken between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, both of which are vertical asymptotes, so that a complete period is shown as a single continuous shape rather than being broken by an asymptote.

Consequently, unlike the other trigonometric graphs, where $\theta = 0$ marks the beginning of each period, for tangent it marks the center of each period, with half the period extending to either side.

Behavior of the Tangent Function over the Reference Interval Over the reference interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the value of $\tan\theta$ may be described in terms of the behavior of $\sin\theta$ and $\cos\theta$.

- At $\theta = -\frac{\pi}{2}$, $\sin\theta = -1$ and $\cos\theta = 0$, so $\tan\theta$ is undefined.
- Over Quadrant IV, $\sin\theta$ is negative and $\cos\theta$ is positive, so $\tan\theta$ is negative.
- On $\left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]$, the magnitude of $\sin\theta$ decreases from 1 to $\frac{\sqrt{2}}{2}$, while the magnitude of $\cos\theta$ increases from 0 to $\frac{\sqrt{2}}{2}$. Thus, $\tan\theta$ decreases in magnitude from ∞ to 1; since the values are negative, this corresponds to an increase from $-\infty$ to -1.
- On $\left[-\frac{\pi}{4}, 0\right]$, the magnitude of $\sin\theta$ decreases from $\frac{\sqrt{2}}{2}$ to 0, while the magnitude of $\cos\theta$ increases from $\frac{\sqrt{2}}{2}$ to 1. Thus, $\tan\theta$ decreases in magnitude from 1 to 0, corresponding to an increase from -1 to 0.
- At $\theta = 0$, $\sin\theta = 0$ and $\cos\theta = 1$, so $\tan\theta = 0$.
- Over Quadrant I, both $\sin\theta$ and $\cos\theta$ are positive, so $\tan\theta$ is positive.
- On $\left[0, \frac{\pi}{4}\right]$, the magnitude of $\sin\theta$ increases from 0 to $\frac{\sqrt{2}}{2}$, while the magnitude of $\cos\theta$ decreases from 1 to $\frac{\sqrt{2}}{2}$. Thus, $\tan\theta$ increases from 0 to 1.
- On $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, the magnitude of $\sin\theta$ increases from $\frac{\sqrt{2}}{2}$ to 1, while the magnitude of $\cos\theta$ decreases from $\frac{\sqrt{2}}{2}$ to 0. Thus, $\tan\theta$ increases from 1 toward ∞.
- At $\theta = \frac{\pi}{2}$, $\sin\theta = 1$ and $\cos\theta = 0$, so $\tan\theta$ is undefined.

Slope Interpretation The value of $\tan\theta$ may also be interpreted as the slope of the terminal line at angle θ. A horizontal line has slope 0, while a vertical line has undefined slope.

Over the reference interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the terminal line rotates through the right half of a circle, from a vertical orientation facing downward to a vertical orientation facing upward. Correspondingly, the value of $\tan\theta$ increases continuously from $-\infty$ to ∞.

- At $\theta = -\frac{\pi}{2}$, the terminal line is vertical and the slope is undefined, so $\tan\theta$ is undefined.
- On $\left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]$, the slope increases from $-\infty$ to -1.
- At $\theta = -\frac{\pi}{4}$, the vertical and horizontal components of the terminal line have equal magnitude, so the slope has magnitude 1 and $\tan\theta = -1$.
- On $\left[-\frac{\pi}{4}, 0\right]$, the slope increases from -1 to 0.
- At $\theta = 0$, the terminal line is horizontal and $\tan\theta = 0$.
- On $\left[0, \frac{\pi}{4}\right]$, the slope increases from 0 to 1.
- At $\theta = \frac{\pi}{4}$, the vertical and horizontal components again have equal magnitude, so $\tan\theta = 1$.
- On $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, the slope increases from 1 toward ∞.
- At $\theta = \frac{\pi}{2}$, the terminal line is vertical and the slope is undefined, so $\tan\theta$ is undefined.

Parity The tangent function is an odd function. Starting from the initial point of the unit circle, $(1, 0)$, positive and negative rotations produce slopes of equal magnitude and opposite sign. Thus,

$$\tan(-x) = -\tan x.$$

Graph of the Basic Function

The preceding paragraphs established the behavior of the values of $\tan\theta$ over the reference interval. Here, the graph of the equation $y = \tan x$ is described.

The graph is described by a single reference period centered at $x = 0$ with length π, spanning the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Over this interval, the graph consists of four equal segments, each of width $\frac{\pi}{4}$.

- At $x = -\frac{\pi}{2}$, $\tan x$ is undefined and there is a vertical asymptote.
- On $\left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]$, $\tan x$ increases from $-\infty$ to -1.
- On $\left[-\frac{\pi}{4}, 0\right]$, $\tan x$ increases from -1 to 0.
- On $\left[0, \frac{\pi}{4}\right]$, $\tan x$ increases from 0 to 1.
- On $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, $\tan x$ increases from 1 toward ∞.
- At $x = \frac{\pi}{2}$, $\tan x$ is undefined and there is a vertical asymptote.

The overall shape of the graph is defined by the vertical asymptotes at the left and right ends of the interval and by three reference points: a left reference point halfway through the left half of the graph at $y = -1$, a center reference point at the horizontal center and vertical midline $(0, 0)$, and a right reference point halfway through the right half of the graph at $y = 1$. The graph rises from $-\infty$ near the left asymptote, passes through these reference points, and continues rising toward ∞ near the right asymptote.

This pattern repeats for every integer n, centered at $x = n\pi$ and spanning the surrounding π interval $\left[n\pi - \frac{\pi}{2},\ n\pi + \frac{\pi}{2}\right]$.

For the basic tangent function, the horizontal center is at $x = 0$, the vertical midline is at $y = 0$, and the vertical stretch factor, which determines the vertical distance of the left and right reference points from the midline, is 1.

Transformations

The graph of $y = \tan x$ may be transformed in the following ways.

For $y = \tan(x + c)$, the graph is shifted horizontally by $-c$. The pattern repeats centered at $-c + \pi n$ instead of πn and spans the interval $\left[-c + \pi n - \frac{\pi}{2},\ -c + \pi n + \frac{\pi}{2}\right]$.

For $y = \tan x + d$, the graph is shifted vertically by d. The vertical midline is located at $y = d$ instead of $y = 0$.

For $y = \tan(bx)$, the graph is compressed horizontally toward the y-axis by a factor of b. The period becomes $\frac{\pi}{b}$. The pattern repeats centered at $\frac{\pi}{b}n$ and spans the surrounding $\frac{\pi}{b}$ interval $\left[\frac{\pi}{b}n - \frac{\pi}{2b},\ \frac{\pi}{b}n + \frac{\pi}{2b}\right]$.

For $y = a(\tan x)$, the graph is stretched vertically from the midline by a factor of $|a|$. The left and right reference points lie a distance $|a|$ below and above the midline.

For $y = \tan(-x)$, the graph is reflected across the x-axis, since the tangent function is odd.

For $y = -\tan x$, the graph is reflected across the x-axis.

Graph of the General Function

For the general tangent function

$$y = a\ \tan(b(x + c)) + d$$

the following properties apply:

- The period is $\frac{\pi}{b}$.
- The phase shift is $-c$.
- Each period is centered at $x = -c + \frac{\pi}{b}n$ and spans the surrounding $\frac{\pi}{b}$ interval $\left[-c + \frac{\pi}{b}n - \frac{\pi}{2b},\ -c + \frac{\pi}{b}n + \frac{\pi}{2b}\right]$.
- Vertical asymptotes occur at the start and end of each period.
- The vertical midline is $y = d$.
- The center reference point is located at the center of the period on the midline, at $\left(-c + \frac{\pi}{b}n,\ d\right)$.
- The left reference point is located $\frac{\pi}{4b}$ to the left of the center and $|a|$ below the midline, and the right reference point is located $\frac{\pi}{4b}$ to the right of the center and $|a|$ above the midline.
- A single period consists of four segments, each of width $\frac{\pi}{4b}$, over which the graph:
  - rises from $-\infty$ to the left reference point,
  - rises from the left reference point to the center reference point,
  - rises from the center reference point to the right reference point,
  - rises from the right reference point toward ∞.
- If a or b is negative, the graph is reflected across the x-axis.

- The domain is all real numbers except at the vertical asymptotes:
  $$\mathsf{Dom}\big(a\ \mathsf{tan}(b(x+c))+d\big)=\{\,x\in\mathbb{R}\mid x\neq -c+\tfrac{\pi}{2b}+\tfrac{k\pi}{b},\ k\in\mathbb{Z}\,\}$$
- The range is all real numbers.

6.7 Graph of the Cotangent Function

Description of the Function

The function $\cot\theta$ is the reciprocal of $\tan\theta$. Its values are related to those of tangent in the same way that the values of cosecant and secant are related to those of sine and cosine, through reciprocal stretching about the vertical stretch lines.

- Where $\tan\theta$ is undefined, $\cot\theta = 0$.
- Where $\tan\theta = \pm 1$, $\cot\theta = \pm 1$.
- Where $\tan\theta = 0$, $\cot\theta$ is undefined.
- As $\tan\theta$ decreases in magnitude from $\pm\infty$ toward ± 1, $\cot\theta$ increases in magnitude from 0 toward ± 1.
- As $\tan\theta$ decreases in magnitude from ± 1 toward 0, $\cot\theta$ increases in magnitude from ± 1 toward $\pm\infty$.

Graph of the Basic Function

Like tangent, the cotangent function has period π.

The reference interval for tangent is taken between the vertical asymptotes at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since cotangent is the reciprocal of tangent, it has a vertical asymptote where $\tan x = 0$. As a result, the reference interval for cotangent is taken beginning at $x = 0$ and spanning the interval $\big[0, \pi\big]$.

Over this interval, the graph consists of four equal segments, each of width $\frac{\pi}{4}$.

- At $x = 0$, $\tan x = 0$, so $\cot x$ is undefined and there is a vertical asymptote.
- On $\big[0, \frac{\pi}{4}\big]$, as $\tan x$ increases from 0 to 1, $\cot x$ decreases from ∞ to 1.
- On $\big[\frac{\pi}{4}, \frac{\pi}{2}\big]$, as $\tan x$ increases from 1 to ∞, $\cot x$ decreases from 1 to 0.
- On $\big[\frac{\pi}{2}, \frac{3\pi}{4}\big]$, as $\tan x$ increases from $-\infty$ to -1, $\cot x$ decreases from 0 to -1.
- On $\big[\frac{3\pi}{4}, \pi\big]$, as $\tan x$ increases from -1 to 0, $\cot x$ decreases from -1 toward $-\infty$.
- At $x = \pi$, $\tan x = 0$, so $\cot x$ is undefined and there is a vertical asymptote.

The overall shape of the graph over this interval is the same as that of tangent over its reference interval, but reflected across the x-axis. It is defined by the vertical asymptotes at the left and right ends of the interval and by three reference points: a center reference point at the horizontal center and vertical midline $\big(\frac{\pi}{2}, 0\big)$, a left reference point halfway through the left half of the graph at $y = 1$, and a right reference point halfway through the right half of the graph at $y = -1$. The graph falls from ∞ near the left asymptote, passes through these reference points, and continues falling toward $-\infty$ near the right asymptote.

This pattern repeats for every integer n, beginning at $x = n\pi$ and spanning the subsequent π interval $\big[n\pi, (n+1)\pi\big]$.

For the basic cotangent function, the horizontal center is at $x = \frac{\pi}{2}$, the vertical midline is at $y = 0$, and the vertical stretch factor, which determines the vertical distance of the left and right reference points from the midline, is 1.

Graph of the General Function

The graph of cotangent is transformed in much the same way as that of tangent. Since cotangent is the reciprocal of tangent, it has the same parity and is an odd function.

For the general cotangent function

$$y = a\ \cot(b(x + c)) + d$$

the following properties apply:

- The period is $\frac{\pi}{b}$.
- The phase shift is $-c$.
- Each period begins at $x = -c + \frac{\pi}{b}n$ and spans the subsequent $\frac{\pi}{b}$ interval $\left[-c + \frac{\pi}{b}n,\ -c + \frac{\pi}{b}(n+1)\right]$.
- Vertical asymptotes occur at the start and end of each period.
- The vertical midline is $y = d$.
- The center reference point is located at the center of the period on the midline, at $\left(-c + \frac{\pi}{b}n + \frac{\pi}{2b},\ d\right)$.
- The left reference point is located $\frac{\pi}{4b}$ to the left of the center and $|a|$ above the midline, and the right reference point is located $\frac{\pi}{4b}$ to the right of the center and $|a|$ below the midline.
- A single period consists of four segments, each of width $\frac{\pi}{4b}$, over which the graph:
  - falls from ∞ to the left reference point,
  - falls from the left reference point to the center reference point,
  - falls from the center reference point to the right reference point,
  - falls from the right reference point toward $-\infty$.
- If a or b is negative, the graph is reflected across the x-axis.
- The domain is all real numbers except at the vertical asymptotes:
  $$\mathrm{Dom}\big(a\ \cot(b(x + c)) + d\big) = \{\ x \in \mathbb{R} \mid x \neq -c + \frac{k\pi}{b},\ k \in \mathbb{Z}\ \}$$
- The range is all real numbers.

Chapter 7

Inverse Trigonometric Functions

A function f maps each input value in its domain to exactly one output value in its range. The inverse function f^{-1} reverses this mapping: for each output value in the range of f, it returns the input value in the domain of f that f maps to that output value.

A function for which no two inputs map to the same output is called **one-to-one**.

While a function may map multiple inputs to the same output, an inverse function can only be defined when each output corresponds to exactly one input. Therefore, only one-to-one functions have defined inverse functions.

The trigonometric functions are periodic, so each trigonometric value corresponds to infinitely many input values. To define inverse trigonometric functions that return a single input value, each inverse function is defined using a specific interval of the corresponding trigonometric function's domain over which the function is one-to-one and each trigonometric value in the range occurs exactly once.

The graph of an inverse trigonometric function may be obtained by reflecting the portion of the corresponding trigonometric graph over this interval across the line $y = x$.

7.1 Inverse Sine Function

The inverse sine function, $\sin^{-1} x$, is defined using the sine function over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This interval corresponds to QIV and QI of the unit circle. Over this interval, every value in the range of $\sin x$, $[-1, 1]$, occurs exactly once.

- At $x = -\frac{\pi}{2}$, $\sin x = -1$.
- On $\left[-\frac{\pi}{2}, 0\right]$, $\sin x$ increases from -1 to 0.
- On $\left[0, \frac{\pi}{2}\right]$, $\sin x$ increases from 0 to 1.
- At $x = \frac{\pi}{2}$, $\sin x = 1$.

The corresponding inverse sine function has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

- At $x = -1$, $\sin^{-1} x = -\frac{\pi}{2}$.
- On $[-1, 0]$, $\sin^{-1} x$ increases from $-\frac{\pi}{2}$ to 0.

- On $[0, 1]$, $\sin^{-1} x$ increases from 0 to $\frac{\pi}{2}$.
- At $x = 1$, $\sin^{-1} x = \frac{\pi}{2}$.

The value of $\sin^{-1} x$ is obtained by determining the input value in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is equal to x.

The inverse sine function is undefined for $x \notin [-1, 1]$.

The identity relationships between $\sin x$ and $\sin^{-1} x$ are:

$$\sin(\sin^{-1} x) = x \quad \text{for } x \in [-1, 1]$$

For $x \notin [-1, 1]$, the inner function is undefined.

$$\sin^{-1}(\sin x) = x \quad \text{for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

For $x \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\sin^{-1}(\sin x)$ returns the value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine equals $\sin x$, rather than x itself.

7.2 Inverse Cosine Function

The inverse cosine function, $\cos^{-1} x$, is defined using the cosine function over the interval $[0, \pi]$. This interval corresponds to $\boldsymbol{QI}$ and $\boldsymbol{QII}$ of the unit circle. Over this interval, every value in the range of $\cos x$, $[-1, 1]$, occurs exactly once.

- At $x = 0$, $\cos x = 1$.
- On $\left[0, \frac{\pi}{2}\right]$, $\cos x$ decreases from 1 to 0.
- On $\left[\frac{\pi}{2}, \pi\right]$, $\cos x$ decreases from 0 to -1.
- At $x = \pi$, $\cos x = -1$.

The corresponding inverse cosine function has domain $[-1, 1]$ and range $[0, \pi]$.

- At $x = -1$, $\cos^{-1} x = \pi$.
- On $[-1, 0]$, $\cos^{-1} x$ decreases from π to $\frac{\pi}{2}$.
- On $[0, 1]$, $\cos^{-1} x$ decreases from $\frac{\pi}{2}$ to 0.
- At $x = 1$, $\cos^{-1} x = 0$.

The value of $\cos^{-1} x$ is obtained by determining the input value in the interval $[0, \pi]$ whose cosine is equal to x.

The inverse cosine function is undefined for $x \notin [-1, 1]$.

The identity relationships between $\cos x$ and $\cos^{-1} x$ are:

$$\cos(\cos^{-1} x) = x \quad \text{for } x \in [-1, 1]$$

For $x \notin [-1, 1]$, the inner function is undefined.

$$\cos^{-1}(\cos x) = x \quad \text{for } x \in [0, \pi]$$

For $x \notin [0, \pi]$, $\cos^{-1}(\cos x)$ returns the value in $[0, \pi]$ whose cosine equals $\cos x$, rather than x itself.

7.3 Inverse Tangent Function

The inverse tangent function, $\tan^{-1} x$, is defined using the tangent function over the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This interval corresponds to $\boldsymbol{QIV}$ and $\boldsymbol{QI}$ of the unit circle. Over this interval, every value in the range of $\tan x$, $(-\infty, \infty)$, occurs exactly once.

- As $x \to -\frac{\pi}{2}^+$, $\tan x \to -\infty$.
- On $\left(-\frac{\pi}{2}, 0\right)$, $\tan x$ increases from $-\infty$ to 0.
- On $\left(0, \frac{\pi}{2}\right)$, $\tan x$ increases from 0 to ∞.
- As $x \to \frac{\pi}{2}^-$, $\tan x \to \infty$.

The corresponding inverse tangent function has domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

- On $(-\infty, 0)$, $\tan^{-1} x$ increases from $-\frac{\pi}{2}$ to 0.
- On $(0, \infty)$, $\tan^{-1} x$ increases from 0 to $\frac{\pi}{2}$.

The value of $\tan^{-1} x$ is obtained by determining the input value in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is equal to x.

The inverse tangent function is defined for all real values of x.

The identity relationships between $\tan x$ and $\tan^{-1} x$ are:

$$\tan(\tan^{-1} x) = x \quad \text{for } x \in \mathbb{R}$$

$$\tan^{-1}(\tan x) = x \quad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

For $x \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tan^{-1}(\tan x)$ returns the value in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent equals $\tan x$, rather than x itself.

Chapter 8

θ as a Parameter of a Solution Space

The trigonometric functions were introduced as assigning sine and cosine values to a specific value of θ; however, when viewed across their full range of outputs and represented graphically, they emerge as a structured system of interrelated values that vary in fixed, repeating patterns.

This system arises from a fundamental constraint on the sine and cosine values:

$$\sin^2\theta + \cos^2\theta = 1$$

The unit circle, defined by the equation $x^2 + y^2 = 1$, represents the complete set of solutions that satisfy this constraint and therefore correspond to valid pairs of sine and cosine values.

From this perspective, the trigonometric values associated with an angle are not primarily functions of the angle itself, but descriptions of a specific point within the solution space determined by this equation. Angular geometry serves as a graphical representation of this mathematical system rather than its underlying foundation.

For this reason, it was emphasized in the previous chapters that under the unit circle method the variable θ represents distance measured along the circumference rather than angle measure alone. Although radian measure corresponds to the same points, θ is treated here not fundamentally as an angle, but as a parameter that indexes position within the solution space defined by the constrained system.

Chapter 9

Trigonometric Identities

9.1 Identities and Equations

In the previous chapters, the trigonometric functions were defined and studied. In this chapter and the next, these functions are applied and manipulated to derive trigonometric identities and to solve trigonometric equations.

A **trigonometric identity** is a relational statement involving one or more variables with specified domains and one or more trigonometric functions, which asserts that the stated relation holds for all values of the variables within those domains.

For example, the statement

$$\sin(-\theta) = -\sin\theta$$

is a trigonometric identity, which asserts that this relation holds for all $\theta \in \mathbb{R}$.

In contrast, a **trigonometric equation** is a relational expression involving one or more variables with specified domains and one or more trigonometric functions that does not assert that the stated relation holds for all values of the variables within those domains.

Solving a trigonometric equation is the process of determining the set of all values of the variable for which the relation is true. This set may consist of all values in the domain, only certain values, or no values at all, in which case the equation has no solution.

For example,

$$\sin\theta = 0$$

is a trigonometric equation. This relation is true only for values of θ that are integer multiples of π.

In contrast, the equation

$$\sin\theta = 3$$

is true for no values of θ and therefore has no solution.

This chapter focuses on the derivation of many trigonometric identities; the next chapter focuses on the methods for solving trigonometric equations.

9.2 Basic Trigonometric Identities

Several trigonometric identities have already been introduced.

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\cot^2 x + 1 = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

Cofunction Identities

Cofunction Identities relate the trigonometric values of complementary angles. On the unit circle, complementary angles are symmetrically located about the line $\theta = \frac{\pi}{4}$ in the first quadrant, so their horizontal and vertical components are exchanged.

$$\sin(\frac{\pi}{2} - x) = \cos x$$
$$\cos(\frac{\pi}{2} - x) = \sin x$$
$$\tan(\frac{\pi}{2} - x) = \cot x$$
$$\cot(\frac{\pi}{2} - x) = \tan x$$
$$\sec(\frac{\pi}{2} - x) = \csc x$$
$$\csc(\frac{\pi}{2} - x) = \sec x$$

9.3 Addition and Subtraction Formulas

The **addition and subtraction formulas** express the trigonometric values of the angles $a+b$ and $a-b$ in terms of the trigonometric values of the angles a and b.

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$
$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$
$$\cos(a-b) = \cos a\cos b + \sin a\sin b$$
$$\sin(a-b) = \sin a\cos b - \cos a\sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$
$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

The derivation of these formulas is based on interpreting angles as rotational operations and examining how successive rotations combine.

Angles as Rotational Operators

Up to this point, angles have been studied as fixed geometric configurations. An angle θ has been associated with a fixed point on the unit circle with coordinates $(\cos\theta, \sin\theta)$, and the trigonometric functions of θ have been identified with these coordinates.

This framework does not provide a direct way to add or subtract angles.

For this purpose, the study of angles shifts from static configurations to the operation of angular rotation, in which an angle θ is examined as the operation of rotating an initial point by an amount θ.

On the unit circle, when $\theta = 0$ and no rotation is applied, the coordinates are $(1, 0)$. In this view, $(1, 0)$ is taken as a normalized base magnitude. The fact that rotation by an angle θ produces the point $(\cos\theta, \sin\theta)$ is interpreted as the effect of the rotational operation by θ acting on a single normalized magnitude, rather than the point itself being the value of θ. When the same rotation is applied to a different initial magnitude, a different resulting configuration is produced.

Rotation as Redistribution of Magnitude Among Dimensions To further develop this view, the value associated with an angle on the unit circle may be regarded as a single magnitude distributed between two orthogonal directions, horizontal and vertical. These directions represent distinct dimensions, and the corresponding horizontal and vertical coordinates are distinct components which together constitute the resulting value.

In this view, the operation of rotation is understood as the redistribution of a fixed total magnitude between the horizontal and vertical orthogonal dimensions. Although the components change, the total magnitude remains unchanged; only its directional distribution changes.

Rotation as an Operation on an Initial Magnitude A normalized single magnitude is defined as one entirely in the default dimension, which by convention is taken to be horizontal and oriented to the right. Thus, the point $(1, 0)$ represents one unit of magnitude in the horizontal direction and zero magnitude in the orthogonal direction.

When an angle operator θ is applied and produces the coordinates $(\cos\theta, \sin\theta)$, this is interpreted as follows: the operation of rotation redistributes the original magnitude into two components, one scaled by $\cos\theta$ and one scaled by $\sin\theta$. The portion scaled by $\cos\theta$ remains in the original orientation, while the portion scaled by $\sin\theta$ is rotated into the orthogonal orientation.

Accordingly, applying an angle θ to the default normalized magnitude results in $\cos\theta$ units of default horizontal magnitude and $\sin\theta$ units of orthogonal vertical magnitude.

The same applies to any other initial magnitude: each component of that magnitude is separately acted upon by the rotation, being scaled by $\cos\theta$ and $\sin\theta$, with the cosine-scaled portion remaining in its original orientation and the sine-scaled portion being rotated into the orthogonal direction.

Orthogonal Rotation To complete this view, orthogonal rotation is described.

In a one-dimensional system, there are two directions: positive and negative. Negating a value reverses the direction of its magnitude within that single dimension.

In an orthogonal system, there are two independent dimensions, and each dimension has two opposing directions, resulting in four distinct directions in total. These may be arranged in a cycle: positive horizontal, positive vertical, negative horizontal, and negative vertical. Progression through this cycle corresponds to successive quarter-turn rotations. Orthogonal rotation is therefore defined as advancing a magnitude one state forward in this cycle.

Importantly, positive and negative here indicate opposing directions within a dimension, not absolute quantities. Accordingly, the transition from positive horizontal to positive vertical is of the same rotational character as the transition from positive vertical to negative horizontal.

Angular Rotation as Orthogonal Rotation The rotational operation of an angle on the portion of magnitude scaled by sine is such an orthogonal rotation: that portion of the magnitude is advanced to the next direction within the ordered four-direction cycle.

Consider the angle $\frac{\pi}{2}$. Its coordinates on the unit circle are $(0, 1)$. This means that a single rotational operation of $\frac{\pi}{2}$ transforms the entire normalized magnitude from the positive horizontal direction to the positive vertical direction. Accordingly, $\cos\frac{\pi}{2} = 0$, since no portion of the original magnitude remains in its original orientation, and $\sin\frac{\pi}{2} = 1$, since the entire magnitude is rotated into the orthogonal direction.

Now consider applying another rotation of $\frac{\pi}{2}$. The resulting angle is π, and the coordinates are $(-1, 0)$. Here, the entire magnitude has been rotated from the positive vertical direction to the negative horizontal direction. Just as the first rotation advanced the magnitude from positive horizontal to positive vertical, this second rotation advances it from positive vertical to negative horizontal, representing a further step through the orthogonal cycle of directions.

A third rotation of $\frac{\pi}{2}$ advances the magnitude from negative horizontal to negative vertical, and a fourth rotation advances it from negative vertical back to positive horizontal.

Thus, repeated orthogonal rotation cycles the magnitude through the four directional states: positive horizontal, positive vertical, negative horizontal, and negative vertical.

Negative Rotation In this view, negative rotation operates in the same manner, except that the portion of the magnitude scaled by sine is advanced backward through the same ordered cycle of orthogonal directions rather than forward.

Thus, instead of progressing from positive horizontal to positive vertical, a negative rotation progresses from positive horizontal to negative vertical, as in the rotation by $-\frac{\pi}{2}$ from $(1, 0)$ to $(0, -1)$.

Derivation of the Addition Formulas for Sine and Cosine

With this framework of angles as operations on a given initial magnitude, the addition and subtraction of angles can now be analyzed.

A rotation by the angle $a + b$ produces the same result as a rotation by a followed by a rotation by b.

Accordingly, $\cos(a+b)$ and $\sin(a+b)$ can be determined by computing the horizontal and vertical components that result from first applying a rotation by a and then applying a rotation by b to a normalized initial magnitude. These components must coincide with the horizontal and vertical components produced by a single rotation through the angle $a + b$, and therefore give the cosine and sine values of $a + b$.

Consider the effect of applying a rotation by a to a normalized initial magnitude.

Beginning with the initial value $(1, 0)$, which consists of one unit of horizontal magnitude and zero vertical magnitude, the rotation by a scales the horizontal magnitude into two parts: $\cos a$ and $\sin a$. The portion scaled by $\cos a$ remains in the original horizontal orientation, while the portion scaled by $\sin a$ is advanced to the next orthogonal direction, here the positive vertical direction, resulting in $\cos a$ units of horizontal magnitude and $\sin a$ units of vertical magnitude.

Now apply a subsequent rotation by the angle b to this result, $(\cos a, \sin a)$. The rotation acts on each component separately.

First, consider the component $\cos a$, which remained in the horizontal orientation after the rotation by a. Applying a rotation by b scales this component into two parts: $\cos a \cos b$ and $\cos a \sin b$. The portion $\cos a \cos b$ is not rotated by b and therefore remains in the positive horizontal direction, while the portion $\cos a \sin b$ is rotated by b into the next orthogonal direction, here the positive vertical direction.

Next, consider the component $\sin a$, which was rotated by a into the positive vertical direction. Applying a rotation by b again scales this component into two parts: $\sin a \cos b$ and $\sin a \sin b$. The portion $\sin a \cos b$ is not rotated by b and therefore remains in the positive vertical direction, while the portion $\sin a \sin b$ is rotated by b into the next orthogonal direction, here from positive vertical to negative horizontal.

Between these four components, the total horizontal magnitude consists of $\cos a \cos b$, which remains positive horizontal, and $\sin a \sin b$, which has been rotated into the negative horizontal direction. The total vertical magnitude consists of $\sin a \cos b$, which was rotated by a but not by b, and $\cos a \sin b$, which was rotated by b but not by a.

The resulting configuration is

$$(\cos a \cos b - \sin a \sin b,\ \sin a \cos b + \cos a \sin b).$$

Since these are the horizontal and vertical components produced by a rotation through the angle $a + b$, it follows that

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

and

$$\sin(a + b) = \sin a \cos b + \cos a \sin b.$$

Derivation of the Subtraction Formulas for Sine and Cosine

The subtraction formulas for sine and cosine are derived in the same manner as the addition formulas, by tracking the effect of a rotation first by a and then by $-b$.

As before, a rotation by a produces $\cos a$ units of horizontal magnitude and $\sin a$ units of vertical magnitude.

Now apply a subsequent rotation by the angle $-b$ to this configuration.

First, consider the component $\cos a$, which remained in the horizontal orientation. Applying a rotation by $-b$ scales this component into two parts: $\cos a \cos b$ and $\cos a \sin b$. The portion $\cos a \cos b$ is not rotated and remains in the positive horizontal direction, while the portion $\cos a \sin b$ is rotated by $-b$ into the previous orthogonal direction, here the negative vertical direction.

Next, consider the component $\sin a$, which was rotated by a into the positive vertical direction. Applying a rotation by $-b$ again scales this component into two parts: $\sin a \cos b$ and $\sin a \sin b$. The portion $\sin a \cos b$ is not rotated and therefore remains in the positive vertical direction, while the portion $\sin a \sin b$ is rotated by $-b$ into the previous orthogonal direction, here from positive vertical back into positive horizontal.

Between these four components, the total horizontal magnitude consists of $\cos a \cos b$, which remains positive horizontal, and $\sin a \sin b$, which has been rotated back into the positive horizontal direction. The total vertical magnitude consists of $\sin a \cos b$, which was rotated by a into the positive vertical direction, and $-\cos a \sin b$, which was rotated by $-b$ into the negative vertical direction.

The resulting configuration is therefore

$$(\cos a \cos b + \sin a \sin b,\ \sin a \cos b - \cos a \sin b).$$

Since these are the horizontal and vertical components produced by a rotation through the angle $a - b$, it follows that

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

and

$$\sin(a-b) = \sin a \cos b - \cos a \sin b.$$

Derivation of the Addition and Subtraction Formulas for Tangent

The addition and subtraction formulas for tangent are derived directly from those of sine and cosine.

By definition,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Substituting the addition formulas for sine and cosine gives

$$\begin{aligned}\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)}\\ &= \frac{\sin a\cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b},\end{aligned}$$

and

$$\begin{aligned}\tan(a-b) &= \frac{\sin(a-b)}{\cos(a-b)}\\ &= \frac{\sin a\cos b - \cos a \sin b}{\cos a \cos b + \sin a \sin b}.\end{aligned}$$

To simplify each expression, divide every term in both the numerator and denominator by $\cos a \cos b$.

The resulting quotients are:

$$\begin{aligned}\frac{\cos a\cos b}{\cos a\cos b} &= 1\\ \frac{\sin a\cos b}{\cos a\cos b} &= \tan a\\ \frac{\cos a\sin b}{\cos a\cos b} &= \tan b\\ \frac{\sin a\sin b}{\cos a\cos b} &= \tan a\tan b.\end{aligned}$$

Applying these substitutions yields

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

and

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}.$$

Rotational Interpretation These formulas may also be interpreted within the rotational framework.

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, tangent may be viewed as the portion of the magnitude that is rotated, normalized by the portion that remains unrotated.

In the addition and subtraction formulas, the factor $\cos a \cos b$ represents the portion of the magnitude that is not rotated by either angle and therefore serves as the normalization factor, reducing to 1.

The remaining terms correspond to rotated components:

- $\tan a$ represents the portion rotated by a,
- $\tan b$ represents the portion rotated by b,
- $\tan a \tan b$ represents the portion rotated by both a and b.

For $a + b$, the rotation produces: a vertical component from a, a vertical component from b, a horizontal component that is not rotated, and a horizontal component rotated by both a and b into the negative direction.

The corresponding tangent value is

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

For $a - b$, the rotation produces: a vertical component from a, a vertical component from $-b$, a horizontal component that is not rotated, and a horizontal component rotated by both angles back into the positive direction.

The corresponding tangent value is

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}.$$

9.4 Double-Angle Formulas

The **double-angle formulas** express the trigonometric functions of the double angle $2x$ in terms of the trigonometric functions of the angle x. These are derived by applying the addition formulas to the case in which an angle is added to itself; for each addition formula, the corresponding double-angle formula is obtained by setting both a and b equal to x.

Double-Angle Formula for Sine

From the addition formula for sine,

$$\sin(a + b) = \sin a \cos b + \cos a \sin b,$$

setting both a and b equal to x gives

$$\sin(2x) = 2 \sin x \cos x.$$

Double-Angle Formula for Cosine

From the addition formula for cosine,

$$\cos(a + b) = \cos a \cos b - \sin a \sin b,$$

setting both a and b equal to x gives

$$\cos(2x) = \cos^2 x - \sin^2 x.$$

Using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, this formula may be written in two additional forms.

Substituting $1 - \sin^2 x$ for $\cos^2 x$ yields

$$\begin{aligned} \cos(2x) &= (1 - \sin^2 x) - \sin^2 x \\ &= 1 - 2 \sin^2 x. \end{aligned}$$

Substituting $1 - \cos^2 x$ for $\sin^2 x$ yields

$$\begin{aligned} \cos(2x) &= \cos^2 x - (1 - \cos^2 x) \\ &= 2 \cos^2 x - 1. \end{aligned}$$

Double-Angle Formula for Tangent

From the addition formula for tangent,

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b},$$

setting both a and b equal to x gives

$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}.$$

Summary of Double-Angle Formulas

$$\sin(2x) = 2\sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$
$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$

9.5 Formulas for Lowering Powers

The **formulas for lowering powers** express the squares of the trigonometric functions of an angle x in terms of the cosine of the double angle $2x$. These formulas are derived from the double-angle formulas for cosine.

Formula for Lowering Powers for Sine

From the double-angle formula

$$\cos 2x = 1 - 2\sin^2 x,$$

solving for $\sin^2 x$ gives

$$\cos 2x = 1 - 2\sin^2 x$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}.$$

Formula for Lowering Powers for Cosine

From the double-angle formula

$$\cos 2x = 2\cos^2 x - 1,$$

solving for $\cos^2 x$ gives

$$\cos 2x = 2\cos^2 x - 1$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}.$$

Formula for Lowering Powers for Tangent

The formula for $\tan^2 x$ is obtained by dividing the expression for $\sin^2 x$ by that for $\cos^2 x$ and canceling the common denominator 2 that appears in both expressions:

$$\begin{aligned}\tan^2 x &= \frac{\sin^2 x}{\cos^2 x}\\ &= \frac{(1-\cos 2x)/2}{(1+\cos 2x)/2}\\ &= \frac{1-\cos 2x}{1+\cos 2x}.\end{aligned}$$

Summary of Formulas for Lowering Powers

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2}\\ \cos^2 x &= \frac{1+\cos 2x}{2}\\ \tan^2 x &= \frac{1-\cos 2x}{1+\cos 2x}\end{aligned}$$

9.6 Half-Angle Formulas

The **half-angle formulas** express the trigonometric functions of $\frac{x}{2}$ in terms of those of x. These formulas are derived from the formulas for lowering powers.

Half-Angle Formula for Sine

The half-angle formula for sine is derived from the formula for lowering powers for sine:

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2}\\ \sin x &= \pm\sqrt{\frac{1-\cos 2x}{2}}\\ \sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}}.\end{aligned}$$

Half-Angle Formula for Cosine

The half-angle formula for cosine is derived from the formula for lowering powers for cosine:

$$\begin{aligned}\cos^2 x &= \frac{1+\cos 2x}{2}\\ \cos x &= \pm\sqrt{\frac{1+\cos 2x}{2}}\\ \cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}}.\end{aligned}$$

For the sine and cosine of $\frac{x}{2}$, the formulas yield an ambiguous sign. Where the value of x is known, the correct sign is determined from the quadrant in which $\frac{x}{2}$ lies.

Half-Angle Formula for Tangent

Deriving the half-angle formula for tangent requires additional steps. First, several identities are established that will be used in the derivation.

From the double-angle formula for sine:

$$\begin{aligned}
\sin 2x &= 2\sin x\cos x\\
\frac{\sin 2x}{2} &= \sin x\cos x\\
\frac{\sin x}{2} &= \sin\frac{x}{2}\cos\frac{x}{2}.
\end{aligned}$$

From the half-angle formula for sine:

$$\begin{aligned}
\sin\frac{x}{2} &= \sqrt{\frac{1-\cos x}{2}}\\
\sin^2\frac{x}{2} &= \frac{1-\cos x}{2}.
\end{aligned}$$

From the half-angle formula for cosine:

$$\begin{aligned}
\cos\frac{x}{2} &= \sqrt{\frac{1+\cos x}{2}}\\
\cos^2\frac{x}{2} &= \frac{1+\cos x}{2}.
\end{aligned}$$

Using these identities, two equivalent half-angle formulas for tangent can be derived.

The first form:

$$\begin{aligned}
\tan\frac{x}{2} &= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\\
&= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\frac{\sin\frac{x}{2}}{\sin\frac{x}{2}}\\
&= \frac{\sin^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}\\
&= \frac{\frac{1-\cos x}{2}}{\frac{\sin x}{2}}\\
&= \frac{1-\cos x}{\sin x}.
\end{aligned}$$

The second form:

$$\begin{aligned}
\tan\frac{x}{2} &= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\\
&= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}\\
&= \frac{\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}\\
&= \frac{\frac{\sin x}{2}}{\frac{1+\cos x}{2}}\\
&= \frac{\sin x}{1+\cos x}.
\end{aligned}$$

These two forms are algebraically equivalent:

$$\begin{aligned}\frac{1-\cos x}{\sin x} &= \frac{\sin x(1-\cos x)}{\sin^2 x}\\ &= \frac{\sin x(1-\cos x)}{(1+\cos x)(1-\cos x)}\\ &= \frac{\sin x}{1+\cos x}.\end{aligned}$$

Summary of Half-Angle Formulas

$$\begin{aligned}\sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}}\\ \cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}}\\ \tan\frac{x}{2} &= \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}\end{aligned}$$

9.7 Product-to-Sum Formulas

The **product-to-sum formulas** express the product of the sine or cosine values of two angles in terms of the sine or cosine of the sum or difference of those angles.

For two angles a and b, there are four possible products:

- $\sin a\cos b$
- $\cos a\sin b$
- $\sin a\sin b$
- $\cos a\cos b$

These products form two pairs, each of which appears in both the addition and subtraction formulas for sine or cosine.

The products $\sin a\cos b$ and $\cos a\sin b$ both appear in the addition and subtraction formulas for sine:

$$\begin{aligned}\sin(a+b) &= \sin a\cos b+\cos a\sin b\\ \sin(a-b) &= \sin a\cos b-\cos a\sin b\end{aligned}$$

The products $\sin a\sin b$ and $\cos a\cos b$ both appear in the addition and subtraction formulas for cosine:

$$\begin{aligned}\cos(a+b) &= \cos a\cos b-\sin a\sin b\\ \cos(a-b) &= \cos a\cos b+\sin a\sin b\end{aligned}$$

For each pair of products, adding the corresponding addition and subtraction formulas cancels one term and isolates the other, while subtracting them does the opposite.

Addition and Subtraction Formulas for Sine Adding the addition and subtraction formulas for sine cancels the $\cos a\sin b$ terms and isolates the $\sin a\cos b$ terms:

$$\begin{aligned}\sin(a+b)+\sin(a-b) &= (\sin a\cos b+\cos a\sin b)+(\sin a\cos b-\cos a\sin b)\\ &= 2\sin a\cos b\\ \frac{\sin(a+b)+\sin(a-b)}{2} &= \sin a\cos b.\end{aligned}$$

Subtracting the subtraction formula for sine from the sine addition formula for sine cancels the $\sin a \cos b$ terms and isolates the $\cos a \sin b$ terms:

$$\begin{aligned}
\sin(a+b) - \sin(a-b) &= (\sin a \cos b + \cos a \sin b) - (\sin a \cos b - \cos a \sin b) \\
&= 2\cos a \sin b \\
\frac{\sin(a+b) - \sin(a-b)}{2} &= \cos a \sin b.
\end{aligned}$$

Addition and Subtraction Formulas for Cosine Adding the addition and subtraction formulas for cosine cancels the $\sin a \sin b$ terms and isolates the $\cos a \cos b$ terms:

$$\begin{aligned}
\cos(a+b) + \cos(a-b) &= (\cos a \cos b - \sin a \sin b) + (\cos a \cos b + \sin a \sin b) \\
&= 2\cos a \cos b \\
\frac{\cos(a+b) + \cos(a-b)}{2} &= \cos a \cos b.
\end{aligned}$$

Subtracting the subtraction formula for cosine from the addition formula for cosine cancels the $\cos a \cos b$ terms and isolates the $\sin a \sin b$ terms:

$$\begin{aligned}
\cos(a+b) - \cos(a-b) &= (\cos a \cos b - \sin a \sin b) - (\cos a \cos b + \sin a \sin b) \\
&= -2\sin a \sin b \\
\frac{\cos(a-b) - \cos(a+b)}{2} &= \sin a \sin b.
\end{aligned}$$

Summary of Product-to-Sum Formulas

$$\begin{aligned}
\sin a \cos b &= \frac{\sin(a+b) + \sin(a-b)}{2} \\
\cos a \sin b &= \frac{\sin(a+b) - \sin(a-b)}{2} \\
\cos a \cos b &= \frac{\cos(a+b) + \cos(a-b)}{2} \\
\sin a \sin b &= \frac{\cos(a-b) - \cos(a+b)}{2}
\end{aligned}$$

9.8 Sum-to-Product Formulas

The **sum-to-product formulas** express the sum or difference of the sine or cosine values of two angles in terms of the sine or cosine of the sum and difference of those angles. These formulas are derived directly from the product-to-sum formulas by reparametrization.

To express the trigonometric values of two angles within a single expression, the angles must be written in relative terms rather than treated as unrelated quantities. This is accomplished by parametrizing the angles using their midpoint and half-difference.

Parametrizing Angles Two angles x and y may be expressed in terms of their midpoint and half-difference.

Let a denote their midpoint, defined by

$$a = \frac{x+y}{2}.$$

Let b denote their half-difference, defined by

$$b = \frac{x-y}{2}.$$

Then the original angles can be written as

$$x = a + b$$

and

$$y = a - b.$$

Substitution into the Product-to-Sum Formulas The terms appearing in the product-to-sum formulas may be interpreted as parametrized forms of two angles x and y. Each occurrence of a and b in the product-to-sum formulas can therefore be rewritten in terms of the original angles x and y.

$$a + b \mapsto x$$

$$a - b \mapsto y$$

$$a \mapsto \tfrac{x+y}{2}$$

$$b \mapsto \tfrac{x-y}{2}.$$

Substitution into

$$\sin a \cos b = \frac{\sin(a+b) + \sin(a-b)}{2}$$

gives

$$\sin\frac{x+y}{2}\cos\frac{x-y}{2} = \frac{\sin x + \sin y}{2}.$$

Substitution into

$$\cos a \sin b = \frac{\sin(a+b) - \sin(a-b)}{2}$$

gives

$$\cos\frac{x+y}{2}\sin\frac{x-y}{2} = \frac{\sin x - \sin y}{2}.$$

Substitution into

$$\cos a \cos b = \frac{\cos(a+b) + \cos(a-b)}{2}$$

gives

$$\cos\frac{x+y}{2}\cos\frac{x-y}{2} = \frac{\cos x + \cos y}{2}.$$

Substitution into

$$\sin a \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}$$

gives

$$\sin\frac{x+y}{2}\sin\frac{x-y}{2} = \frac{\cos y - \cos x}{2}.$$

Final Derivation Multiplying both sides of each substituted equation by 2 and rewriting the result in terms of the variables a and b yields the sum-to-product formulas:

$$2\sin\frac{a+b}{2}\cos\frac{a-b}{2} = \sin a + \sin b$$

$$2\cos\frac{a+b}{2}\sin\frac{a-b}{2} = \sin a - \sin b$$

$$2\cos\frac{a+b}{2}\cos\frac{a-b}{2} = \cos a + \cos b$$

$$2\sin\frac{a+b}{2}\sin\frac{a-b}{2} = \cos b - \cos a$$

Chapter 10

Solving Trigonometric Equations

10.1 Basic Trigonometric Equations

The most basic trigonometric equation involving a trigonometric function f has the form

$$f(\theta) = c$$

where the value c is given. Solving the equation consists of determining all values of θ for which the relation holds.

More complicated trigonometric equations are solved by algebraic manipulation that reduces them to this basic form.

For example, to solve

$$\frac{\cos\theta + 3}{2} = 2,$$

first simplify the equation to obtain

$$\cos\theta = 1,$$

and then determine all values of θ for which this equation is true.

Periodic Values

Since the trigonometric functions are periodic, any trigonometric equation with no specified domain that has at least one solution has infinitely many solutions. For example, the equation

$$\cos\theta = 1$$

holds at $\theta = 0,\ 2\pi,\ 4\pi,\ \ldots$.

The **principal solution** of a trigonometric equation is the solution for θ within the interval $[0, 2\pi)$. For $\cos\theta = 1$, the principal solution is $\theta = 0$.

The **complete solution** represents the full set of all solutions. For $\cos\theta = 1$, the complete solution is expressed as

$$\theta = 2k\pi \text{ for any integer } k.$$

In practice, the phrase "for any integer k" is often omitted, and it is understood that k represents an arbitrary integer.

For equations that reduce to this basic form, each solution of a sine or cosine equation involving a nonquadrantal angle has the form

$$\theta = \theta_0 + 2k\pi,$$

since the period of sine and cosine is 2π.

Each solution of a tangent equation has the form

$$\theta = \theta_0 + k\pi,$$

since the period of tangent is π.

Solutions Within a Period

For basic sine and cosine equations that do not involve quadrantal angles, there are two solutions within a single period.

For example, the equation

$$\cos\theta = \frac{1}{2}$$

holds at both $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

The **principal solutions** for this equation are therefore

$$\left\{ \frac{\pi}{3}, \frac{5\pi}{3} \right\},$$

and the **complete solution** is

$$\left\{ \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi \right\}.$$

For sine equations, the two solutions within a single period are symmetric across the y-axis, occurring in $\boldsymbol{QI}$ and $\boldsymbol{QII}$ or in $\boldsymbol{QIII}$ and $\boldsymbol{QIV}$. For cosine equations, the two solutions within a single period are symmetric across the x-axis, occurring in $\boldsymbol{QI}$ and $\boldsymbol{QIV}$ or in $\boldsymbol{QII}$ and $\boldsymbol{QIII}$.

Quadrantal Angles

Equations that involve quadrantal angles are exceptions to the preceding rules.

When the trigonometric value is 1 or -1, there is only one solution within each 2π interval, rather than two:

- $\sin\theta = 1$ at $\theta = \frac{\pi}{2} + 2k\pi$,
- $\sin\theta = -1$ at $\theta = \frac{3\pi}{2} + 2k\pi$,
- $\cos\theta = 1$ at $\theta = 2k\pi$,
- $\cos\theta = -1$ at $\theta = \pi + 2k\pi$.

When the trigonometric value is 0, the value repeats every half rotation, so the period is π, rather than 2π:

- $\sin\theta = 0$ at $\theta = k\pi$,
- $\cos\theta = 0$ at $\theta = \frac{\pi}{2} + k\pi$.

Equations with No Solution The ranges of the sine and cosine functions are $[-1, 1]$. If a trigonometric equation specifies a value outside this interval, the equation has no solution.

For example, the equation

$$\sin\theta = 2$$

has no solution.

10.2 Complex Trigonometric Equations

More complex trigonometric equations often require intermediate algebraic steps to reduce them to a basic trigonometric equation.

The following examples illustrate common techniques used in this process.

Taking the Square Root

For an equation involving a squared trigonometric function, such as

$$\sin^2\theta = \frac{3}{4},$$

take the square root of both sides:

$$\sin\theta = \pm\frac{\sqrt{3}}{2}.$$

This equation has four principal solutions:

$$\theta = \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3}.$$

Factoring Quadratics

For an equation involving both a linear and a squared trigonometric term, such as

$$2\sin^2\theta - 3\sin\theta + 1 = 0,$$

factor the quadratic:

$$(2\sin\theta - 1)(\sin\theta - 1) = 0.$$

By the zero factor rule, at least one of the factors must be zero. This splits into two equations:

$$2\sin\theta - 1 = 0$$
$$\sin\theta = \frac{1}{2},$$

or

$$\sin\theta - 1 = 0$$
$$\sin\theta = 1.$$

The first equation has two principal solutions, and the second has one principal solution. Altogether, this equation has three principal solutions:

$$\theta = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{\pi}{2}.$$

Replacing Using the Pythagorean Identity

For an equation involving both $\sin\theta$ and $\cos\theta$, where at least one term is squared, use a Pythagorean identity to replace one term so that only one trigonometric function remains.

For example, to solve

$$\sin^2\theta = 5 - 2\cos^2\theta,$$

rewrite $\sin^2\theta$ using the identity $\sin^2\theta = 1 - \cos^2\theta$:

$$1 - \cos^2\theta = 5 - 2\cos^2\theta.$$

Then simplify:

$$\begin{aligned} 1 - \cos^2\theta &= 5 - 2\cos^2\theta \\ \cos^2\theta &= 4 \\ \cos\theta &= \pm 2. \end{aligned}$$

Since $\cos\theta$ has range $[-1, 1]$, this equation has no solution.

Reducing Reciprocal Functions

For an equation containing a single trigonometric function $\sec\theta$, $\csc\theta$, or $\cot\theta$, first reduce to a basic trigonometric equation in that function. Then convert to an equivalent equation in $\sin\theta$, $\cos\theta$, or $\tan\theta$.

For example, to solve

$$3\sec^2\theta - 4 = 0,$$

first reduce to a basic trigonometric equation in secant:

$$\begin{aligned} 3\sec^2\theta &= 4 \\ \sec^2\theta &= \frac{4}{3} \\ \sec\theta &= \pm\frac{2}{\sqrt{3}}. \end{aligned}$$

Then convert to an equivalent equation in cosine using the reciprocal identity:

$$\cos\theta = \pm\frac{\sqrt{3}}{2}.$$

Solving for a Composite Input

The input to a trigonometric function is **composite** if it is an expression in θ rather than θ alone. For example, the input to $\sin(3\theta)$ is composite, since 3θ is an expression in θ.

For a trigonometric equation with a composite input, first find the complete solution treating the input as a simple variable. Then replace the variable with the given expression and solve.

For example, to solve

$$\cos(3\theta) = \frac{1}{2},$$

first solve the equation

$$\cos x = \frac{1}{2}.$$

This yields the complete solution

$$x = \frac{\pi}{6} + 2k\pi,\ \frac{11\pi}{6} + 2k\pi.$$

Then substitute x with 3θ and solve for θ:

$$\begin{aligned} 3\theta &= \frac{\pi}{6} + 2k\pi \\ \theta &= \frac{\pi}{18} + \frac{2k\pi}{3}, \end{aligned}$$

or

$$\begin{aligned} 3\theta &= \frac{11\pi}{6} + 2k\pi \\ \theta &= \frac{11\pi}{18} + \frac{2k\pi}{3}. \end{aligned}$$

The resulting complete solution set is

$$\theta = \frac{\pi}{18} + \frac{2k\pi}{3},\ \frac{11\pi}{18} + \frac{2k\pi}{3}.$$

Simplifying Using Trigonometric Identities

Certain composite inputs are simplified using trigonometric identities.

For example, to solve

$$\cos(2\theta) = 5\sin\theta - 2,$$

apply the double-angle identity

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

and substitute:

$$\cos^2\theta - \sin^2\theta = 5\sin\theta - 2.$$

Rewrite $\cos^2\theta$ using the Pythagorean identity and simplify:

$$1 - \sin^2\theta - \sin^2\theta = 5\sin\theta - 2$$

$$2\sin^2\theta + 4\sin\theta - 3 = 0.$$

This equation is solved as a quadratic in $\sin\theta$.

Chapter 11

Laws of Sines and Cosines

11.1 Trigonometric Laws Applicable to All Triangles

In the second chapter, the trigonometric functions were defined for an acute angle in a right triangle as relationships among the lengths of its sides. Using these definitions, the trigonometric functions were applied to solve right triangles as functions of an acute angle.

In the later chapters, the trigonometric functions have been extended beyond triangles to any angle in standard position, where sine and cosine are defined as the horizontal and vertical components of the terminal ray. Although under these definitions the trigonometric functions are not defined directly in terms of triangles, their properties as functions of angles give rise to laws that apply to all triangles. These are the **Law of Sines** and the **Law of Cosines**.

Law of Sines

The Law of Sines states that for a triangle with sides a, b, and c, and corresponding opposite angles A, B, and C, the ratios of the side lengths are proportional to the sines of their opposite angles:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Law of Cosines

The Law of Cosines relates the lengths of the sides of a triangle to the cosine of one of its angles. For a triangle with sides a, b, and c, and angle C opposite side c:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Pythagorean Theorem The Pythagorean Theorem is a special case of the more general Law of Cosines applied to right triangles: if angle C is a right angle, then $\cos 90^\circ = 0$, and the Law of Cosines reduces to:

$$c^2 = a^2 + b^2$$

for the hypotenuse.

11.2 Solving Oblique Triangles

An **oblique triangle** is a triangle that does not contain a right angle. The Law of Sines and the Law of Cosines may be used to solve oblique triangles.

Solving a triangle means determining the measures of its three sides and its three angles. To solve a triangle, at least three of these six quantities must be known, with at least one of them being a side length. If no side lengths are known, there is no way to determine any of the side lengths, even if all three angles are known.

When three quantities are known and at least one is a side length, there are three possible sets of given information: two angles and one side, one angle and two sides, or three sides.

Two Angles and One Side

If two angles and one side are known, the third angle is found by subtracting the sum of the first two from 180°, since the interior angles of a triangle always sum to 180°. Once three angles and one side are known, the remaining two sides can be found using the Law of Sines.

Two Sides and One Angle

If two sides and one angle are known, there are two cases. The known angle may be the angle between the two sides, or the angle adjacent to one side and opposite the other. The first case is referred to as **side–angle–side** (SAS), and the second as **side–side–angle** (SSA).

SAS

For SAS, the third side is found using the Law of Cosines. Once all three sides and one angle are known, the remaining two angles may be determined using either the Law of Sines or the Law of Cosines.

SSA

The SSA case is called the **ambiguous case** because it may have zero, one, or two solutions, depending on the given values.

Let the known angle be A, the known side opposite A be a, and the known side adjacent to A be b. Applying the Law of Sines to determine B gives:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\sin B = \frac{b \sin A}{a}.$$

In a valid triangle, all side lengths are positive, and each interior angle lies in the interval $(0, 180°)$, with a corresponding sine value in the range $(0, 1]$. Thus, for any valid set of given values, the computed value of $\sin B$ must be positive.

Analysis of $\sin B$ If $\frac{b \sin A}{a} > 1$, then no triangle exists, since no angle has a sine value greater than 1. Thus, if $b \sin A > a$, there is no solution.

If $\frac{b \sin A}{a} = 1$, then B must be a right angle, since $\sin B = 1$ only when $B = 90°$. Thus, if $b \sin A = a$, there is exactly one solution.

If $0 < \frac{b \sin A}{a} < 1$, then $\sin B$ lies strictly between 0 and 1. In this range, a given sine value corresponds to two possible angles, one acute and one obtuse, so the value of B cannot be uniquely determined from $\sin B$ alone.

Where $0 < \frac{b \sin A}{a} < 1$, further analysis is required to determine whether zero, one, or two valid triangles exist.

Analysis by Measure of Angle A Orient the triangle in the coordinate plane so that vertices A and B lie at the same vertical coordinate line, with B to the right of A, and vertex C lying above A and B. In this orientation, side c lies horizontally between A and B, while sides b and a extend upward from A and B, respectively, and meet at C.

The horizontal position of C relative to A depends on the measure of angle A:

- If A is obtuse, then side b must extend upward and to the left from A, since side c extends horizontally to the right. Thus, the endpoint C of side b lies to the left of A.
- If A is a right angle, then side b extends vertically upward from A, and C lies directly above A.
- If A is acute, then side b extends upward and to the right from A, placing C to the right of A.

Similarly, the horizontal position of C relative to B depends on the measure of angle B.

- If B is obtuse, then side a must extend upward and to the right from B, and the endpoint C lies to the right of B.
- If B is a right angle, then side a extends vertically upward from B, and C lies directly above B.
- If B is acute, then side a extends upward and to the left from B, and C lies to the left of B.

From this analysis, the following conclusion can be drawn:

If A is obtuse or a right angle, then C lies on or to the left of A. Since B lies to the right of A, it follows that C must lie to the left of B. Therefore, angle B cannot be obtuse in this case.

Thus, when A is obtuse or right, of the two angles corresponding to the computed value of $\sin B$, only the acute angle produces a valid triangle.

Analysis by Lengths of Sides a and b Further analysis involves comparing the relative lengths of sides a and b to determine the horizontal ordering of the points A, B, and C. This comparison is based on the fact that sides a and b extend from the common point C to B and A, respectively, and that B lies to the right of A.

If A is obtuse or a right angle, then C lies on or to the left of A. In this case, the segment from C to B must extend farther horizontally than the segment from C to A, so side a must be longer than side b.

Similarly, if B is obtuse or a right angle, then C lies on or to the right of B. In this case, the segment from C to A must extend farther horizontally than the segment from C to B, so side b must be longer than side a.

From this analysis, the following conclusions can be drawn:

- If A is right or obtuse, a triangle is possible only if $a > b$. In this case, B must be acute, as previously established. Thus, when A is right or obtuse:
  - if $a \leq b$, there is no solution, and
  - if $a > b$, there is exactly one solution.
- If A is acute, both the acute and obtuse options for B are initially possible. However, the obtuse option for B is only possible when $a < b$. If $a > b$, the obtuse configuration is impossible, and only the acute value of B produces a valid triangle. Thus, when A is acute:

  - If $a < b$, there are two solutions, and
  - if $a \geq b$, there is exactly one solution.

Summary of Rules The complete series of rules for the SSA case may be summarized as follows. The value of $\sin B$ is determined by

$$\sin B = \frac{b \sin A}{a}.$$

- If $\frac{b \sin A}{a} > 1$, there is no solution.
- If $\frac{b \sin A}{a} = 1$, there is exactly one solution, and B is a right angle.
- If $0 < \frac{b \sin A}{a} < 1$, then $\sin B$ corresponds to two possible angles, one acute and one obtuse, and further analysis is required:
  - If A is right or obtuse:
    * If $a \leq b$, there is no solution.
    * If $a > b$, there is exactly one solution, and B is the acute angle corresponding to this sine value.
  - If A is acute:
    * If $a < b$, there are two solutions, corresponding to B being acute or obtuse.
    * If $a \geq b$, there is exactly one solution, and B is the acute angle corresponding to this sine value.

Once angle B is determined, angle C is found by subtracting A and B from 180°, and the remaining side c is found using either the Law of Sines or the Law of Cosines. Where there are two valid values of B, there are two corresponding values of C and c.

Three Sides

If all three sides are given, the Law of Cosines may be used to find any angle:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\ C &= \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right). \end{aligned}$$

The remaining angles may then be found using the same method or by the Law of Sines.